Manin matrices and Talalaev's formula

2008 J. Phys. A: Math. Theor. 41 194006

(http://iopscience.iop.org/1751-8121/41/19/194006)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **41** (2008) 194006 (28pp) doi:10.1088/1751-8113/41/19/194006

Manin matrices and Talalaev's formula

A Chervov and G Falqui

Institute for Theoretical and Experimental Physics, Università degli Studi di Milano, Bicocca, Italy

E-mail: chervov@itep.ru and gregorio.falqui@unimib.it

Received 6 February 2008, in final form 7 April 2008
Published 29 April 2008
Online at stacks.iop.org/JPhysA/41/194006

Abstract

In this paper we study properties of Lax and transfer matrices associated with quantum integrable systems. Our point of view stems from the fact that their elements satisfy special commutation properties, considered by Yu I Manin some 20 years ago at the beginning of quantum group theory. These are the commutation properties of matrix elements of linear homomorphisms between polynomial rings; more explicitly these read: (1) elements of the same column commute; (2) commutators of the cross terms are equal: $[M_{ij}, M_{kl}] = [M_{kj}, M_{il}]$ (e.g. $[M_{11}, M_{22}] = [M_{21}, M_{12}]$). The main aim of this paper is twofold: on the one hand we observe and prove that such matrices (which we call *Manin* matrices in short) behave almost as well as matrices with commutative elements. Namely, the theorems of linear algebra (e.g., a natural definition of the determinant, the Cayley–Hamilton theorem, the Newton identities and so on and so forth) have a straightforward counterpart in the case of Manin matrices. On the other hand, we remark that such matrices are somewhat ubiquitous in the theory of quantum integrability. For instance, Manin matrices (and their q-analogs) include matrices satisfying the Yang–Baxter relation 'RTT=TTR' and the so-called Cartier–Foata matrices. Also, they enter Talalaev's remarkable formulae: $\det(\partial_z - L_{\mathrm{Gaudin}}(z))$, $\det(1 - e^{-\partial_z} T_{\mathrm{Yangian}}(z))$ for the 'quantum spectral curve', and appear in the separation of variables problem and Capelli identities. We show that theorems of linear algebra, after being established for such matrices, have various applications to quantum integrable systems and Lie algebras, e.g. in the construction of new generators in $Z(U_{\mathrm{crit}}(\widehat{gl_n}))$ (and, in general, in the construction of quantum conservation laws), in the Knizhnik–Zamolodchikov equation, and in the problem of Wick ordering. We propose, in the appendix, a construction of quantum separated variables for the XXX-Heisenberg system.

PACS numbers: 02.30.Ik, 02.20.Uw

1. Introduction and summary of the results

It is well known that matrices with generically noncommutative elements do not admit a natural construction of the determinant, and basic theorems of the linear algebra fail to hold true. On the other hand, matrices with noncommutative entries play a basic role in the theory of quantum integrability (see, e.g., [FT79], in Manin's theory of 'noncommutative symmetries' [Man88], and so on and so forth. It is fair to say that recently D Talalaev [Ta04] made a kind of breakthrough in quantum integrability, defining the 'quantum characteristic polynomial' or 'quantum spectral curve' $\det(\partial_z - L(z))$ for Lax matrices satisfying rational R-matrix commutation relations (e.g., Gaudin systems)[1].

The first and basic observation of the present paper is the following: the *quantum Lax matrix* $(\partial_z - L_{gl_n-\text{Gaudin}}(z))$ entering[2] Talalaev's formula, as well as a suitable modification of the transfer matrix of the (XXX) Heisenberg chain matches the simplest case of Manin's considerations (i.e. are Manin matrices in the sense specified by the definition 1). Further we prove that many *results of commutative linear algebra* can be applied with minor modifications in the case of 'Manin matrices' and derive applications.

We will consider the simplest case of those considered by Manin, namely—in the present paper—we will restrict ourselves to the case of commutators, and not of (super)-q-commutators, etc. Let us mention that Manin matrices are defined, roughly speaking, by imposing half of the relations of the corresponding quantum group $Fun_q(GL(n))$ and taking $q = 1$ (see remark 2).

Definition 1. *Let M be a $n \times n'$ matrix with elements M_{ij} in a noncommutative ring $\mathcal{R}$. We will call M a Manin matrix if the following two conditions hold:*

(1) elements in the same column commute between themselves.
(2) commutators of cross terms of 2×2 submatrices of M are equal:

$$[M_{ij}, M_{kl}] = [M_{kj}, M_{il}], \qquad \forall\, i, j, k, l \quad \textit{e.g.} \quad [M_{11}, M_{22}] = [M_{21}, M_{12}]. \tag{1}$$

A more intrinsic definition of Manin matrices via coactions on polynomial and Grassmann algebras will be given in the proposition 1.

As we shall see in section 3, the following properties hold:

- $\partial_z - L_{gl_n-\text{Gaudin}}(z)$ is a Manin matrix, where $L_{gl_n-\text{Gaudin}}(z)$ is the Lax matrix for the Lie algebra $gl_n[t]$ (section 3.1).
- $\mathrm{e}^{-\partial_z} T_{gl_n-\text{Yangian}}(z)$ is a Manin matrix, where $T_{gl_n-\text{Yangian}}(z)$ is the Lax (or 'transfer') matrix for the Yangian algebra $Y(gl_n)$ (section 3.2).

Furthermore, we shall see that Manin matrices enter other topics in quantum integrability, e.g., the quantum separation of variables theory (appendix A.2) and Capelli identities (section 4.3.1).

Among the basic statements of linear algebra we establish (in the form *identical* to the commutative case) for Manin matrices, let us mention:

- section 4.2: the inverse of a Manin matrix M is again Manin;
- section 4.3: the formula for the determinant of block matrices:

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A)\det(D - CA^{-1}B) = \det(D)\det(A - BD^{-1}C);$$

[1] See [CT06, MTV06, MTV05] for some applications to the Bethe ansatz, separation of variables, the Langlands correspondence, the Capelli identities, real algebraic geometry and related topics.

[2] Remark that $L_{gl_n-\text{Gaudin}}(z)$ is not a Manin matrix; to insert ∂_z is an insight due to D Talalaev; the operator $(\partial_z - L_{gl_n-\text{Gaudin}}(z))$ is also related to the Knizhnik–Zamolodchikov equation, see section 4.1.1.

- section 5.1: the Cayley–Hamilton theorem: $\det(t - M)|_{t=M} = 0$;
- section 5.2: the Newton identities between $\mathrm{Tr}\, M^k$ and coefficients of $\det(t + M)$).

This extends some results previously obtained in the literature: Manin has defined the determinant, proved Cramer's inversion rule, Laplace formulae, as well as Plucker identities. In [GLZ03] the MacMahon–Wronski formula was proved; [Ko07B, Ko07A] contains Sylvester's identity and the Jacobi ratio's theorem, along with partial results on an inverse matrix and block matrices[3].

One of our point is to present applications to the theory of quantum integrable systems, e.g., to the Knizhnik–Zamolodchikov equation, as well as Yangians and Lie algebras. Namely:

- section 4.1.1: a new proof of the correspondence [CT04] between solutions of the Knizhnik–Zamolodchikov equation and solutions $Q(z)$ of $\det(\partial_z - L(z))Q(z) = 0$.
- section 4.2.1: the observation that the inverse of a Manin matrix is again a Manin matrix yields, in a very simple way, some recent result on a general construction about quantum integrable systems [ER01, BT02], that, in a nutshell, says that (quantum)separability implies commutativity of the quantum Hamiltonians.
- section 4.3.1: a new proof of the generalized Capelli identities [MTV06] for the Lie algebra $gl[t]$ (or 'the Gaudin integrable system').
- section 5.2.1: the construction of new explicit generators ($: \mathrm{Tr}\, L^{[k]}(z) :$)[4] in the center of $U_{c=\text{crit}}(gl_n[t, t^{-1}], c)$ and ($\mathrm{Tr}\, L^{[k]}(z)$) in the commutative Bethe subalgebra in $U(gl_n[t])$.
- section 5.3.1: the construction of a further set of explicit generators ($\mathrm{Tr}\, S^k L(z)$) in the commutative Bethe subalgebra in $U(gl_n[t])$ via a 'quantum' MacMahon–Wronski formula.

In the appendix, we collect definitions about the center of $U_{c=\text{crit}}(gl_n[t, t^{-1}], c)$, that is, the commutative Bethe subalgebra in $U(gl[t])$ (appendix A.1), and, finally we address the problem of Sklyanin's separation of variables problem; we present a conjectural construction of the quantum separated variables for Yangian-type systems.

Our starting point is the following remarkable construction by D Talalaev of the 'quantum spectral curve' or the 'quantum characteristic polynomial'. It solved the long standing problems of the explicit efficient construction of the center of $U_{c=\text{crit}}(gl_n[t, t^{-1}], c)$, the commutative Bethe subalgebra in $U(gl[t])$[5], and explicitly produces a complete set of quantum integrals of motion for the Gaudin system. As we shall see, it also has many other applications. The right setup where Talalaev's formula fits is the ideas of E K Sklyanin on quantum separation of variables (see, e.g., the surveys [Sk92, Sk95]).

Theorem [Ta04]**.** *Let $L(z)$ be the Lax matrix of the gl_n-Gaudin model, and consider the following differential operator in the variable z (Talalaev's quantum characteristic polynomial or quantum spectral curve):*

$$\det{}^{\text{column}}(\partial_z - L_{gl_n-\text{Gaudin}}(z)) = \sum_{i=0,\dots,n} QH_i(z)\partial_z^i, \tag{2}$$

Then

$$\forall\, i, j \in 0, \dots, n, \quad \text{and} \quad u \in \mathbb{C}, v \in \mathbb{C} \quad [QH_i(z)|_{z=u}, QH_j(z)|_{z=v}] = 0.$$

[3] These authors actually considered more general classes of matrices.
[4] The basic definitions on 'normal ordering' $: \cdots :$, 'critical level', $U_{c=\text{crit}}(\widehat{gl_n}) = U_{c=\text{crit}}(gl_n[t, t^{-1}], c)$, and ('Bethe') subalgebra in $U(gl_n[t])$) are recalled in appendix A.1.
[5] See [CT06, CM].

So taking $QH_i(z)$ for different $i, z = u \in \mathbb{C}$ (or their residues at poles, or other 'spectral invariant' of the Lax matrix $\partial_z - L_{gl_n-\text{Gaudin}}(z)$) one obtains a full set of generators of quantum mutually commuting conserved integrals of motion.

Actually, the theorem holds for all Lax matrices of "gl_n-Gaudin type (see definition 4, section 3.1).

For the Yangian case D Talalaev considers: $\det(1 - e^{-\partial_z} T_{gl_n-\text{Yangian}}(z))$ ([Ta04] formula (9) page 6).

Remarks

(1) In [GLZ03, Ko07B, Ko07A] the name 'right quantum matrices' was used. We prefer to use the name 'Manin matrices'.
(2) Our case is different from the more general one of [GGRW02], where *generic* matrices with noncommutative entries are considered. In this case, there is no natural definition of the determinant, and the analogues of linear algebra propositions are sometimes quite different from the commutative case. Nevertheless some results of the above-mentioned authors can be fruitfully applied to some questions here.
(3) Those readers that are familiar with the R-matrix approach to (quantum) integrable systems know that $L_{gl_n-\text{Gaudin}}(z)$ and $T_{gl_n-\text{Yangian}}(z)$ satisfy quite *different* commutation relations (9), (16) with 'spectral parameter', namely, in the first case we have linear R-matrix commutation relations, while in the second one we have quadratic relations. It is quite surprising that Talalaev's introduction of ∂_z converts both to the *same* class of Manin matrices.

Moreover Manin's relations do *not* contain explicitly the *'spectral parameter'* z. Thus we can use simpler considerations (that is, without taking into account the dependence on spectral parameter) and apply our results also to the theory z-dependent Lax/transfer matrices. Another feature of insertion of ∂_z is that the somewhat *ad hoc shifts* in the spectral parameter entering the formulae for the '$q\det(T(z))$' known in the literature now *appear automatically* from the column-determinant expansion of Manin matrices.

All the considerations below work for an arbitrary field of characteristic not equal to 2, but we restrict ourselves to $\mathbb{C}$.

2. Manin matrices: definitions and elementary properties

We herewith recall definitions and results of [Man87, Man88, Man91] (with minor variations suited to the sequel of the paper). We first remark that the notion (given in definition 1) of Manin matrix with elements in an associative ring $\mathcal{R}$ can be reformulated as the condition that

$$\forall\, p, q, k, l [M_{pq}, M_{kl}] = [M_{kq}, M_{pl}] \text{ e.g.}$$
$$[M_{11}, M_{22}] = [M_{21}, M_{12}], [M_{11}, M_{2k}] = [M_{21}, M_{1k}].$$

Indeed, for $q = l$ the above requirement yields that elements that belong to the same column of M commute among themselves. For $q \neq l$, it is precisely the second condition of definition 1.

Remark 1. Precisely these relations were explicitly written in [Man88] (see chapter 6.1, especially formula (1)). Implicitly they are contained in [Man87]—the last sentence on page 198 contains a definition of the algebra $end(A)$ for an arbitrary quadratic Koszul algebra A. One can show (see the remarks on page 199, top) that $end(\mathbb{C}[x_1, \ldots, x_n])$ is the algebra generated by M_{ij}.

Remark 2. *Relation with quantum group $Fun_q(GL(n))$* (see [Man88, CFRu]). Let us consider $n = 2$. The q-analog of the above relations is: $M_{21}M_{11} = qM_{11}M_{21}$, $M_{22}M_{12} = qM_{12}M_{22}$, $M_{11}M_{22} = M_{22}M_{11}+q^{-1}M_{21}M_{12}-qM_{12}M_{21}$, this is precisely half of the relations defining the quantum group $Fun_q(GL(2))$ (e.g. [Man87], page 192, formula (3)). This is true in general: q-Manin matrices are obtained by imposing half of the relations of the corresponding quantum group $Fun_q(GL(n))$. Conversely, one can define $Fun_q(GL(n))$ by matrices M such that M and M^t are simultaneously q-Manin matrices [Man87, Man88]. In the present paper we consider the $q = 1$ case.

A matrix M such that M^t is a Manin matrix satisfies analogous 'good' properties as Manin matrices do. We will point out this case explicitly when needed.

The proposition below gives a more intrinsic characterization of Manin's matrices ([Man87], page 199 (top), [Man88, Man91]):

Proposition 1. *Consider a rectangular $n\times m$-matrix M, the polynomial algebra $\mathbb{C}[x_1,\ldots,x_m]$ and the Grassman algebra $\mathbb{C}[\psi_1,\ldots,\psi_n]$ (i.e. $\psi_i^2 = 0$, $\psi_i\psi_j = -\psi_j\psi_i$); suppose x_i and ψ_i commute with the matrix elements $M_{p,q}$. Consider the variables $\tilde{x}_i$ and $\tilde{\psi}_i$ defined by:*

$$\begin{pmatrix}\tilde{x}_1\\ \cdots\\ \tilde{x}_n\end{pmatrix} = \begin{pmatrix}M_{1,1} & \cdots & M_{1,m}\\ \cdots & & \\ M_{n,1} & \cdots & M_{n,m}\end{pmatrix}\begin{pmatrix}x_1\\ \cdots\\ x_m\end{pmatrix},$$
$$(\tilde{\psi}_1,\ldots,\tilde{\psi}_m) = (\psi_1,\ldots,\psi_n)\begin{pmatrix}M_{1,1} & \cdots & M_{1,m}\\ \cdots & & \\ M_{n,1} & \cdots & M_{n,m}\end{pmatrix}, \tag{3}$$

that is the new variables are obtained via the left action (in the polynomial case) and the right action (in the Grassmann case) of M on the old ones. Then the following three conditions are equivalent:

- *the matrix M is a Manin matrix;*
- *the variables $\tilde{x}_i$ commute among themselves: $[\tilde{x}_i,\tilde{x}_j] = 0$;*
- *the variables $\tilde{\psi}_i$ anticommute among themselves: $\tilde{\psi}_i\tilde{\psi}_j + \tilde{\psi}_j\tilde{\psi}_i = 0$.*

Proof. It is a straightforward calculation. □

Let us present some examples of Manin matrices.

Definition 2. *A matrix A with the elements in a noncommutative ring is called a Cartier–Foata (see [CF69, Fo79]) matrix if elements from different rows commute with each other.*

Lemma 1. *Any Cartier–Foata matrix is a Manin matrix.*

Proof. The characteristic conditions of definition 1 are trivially satisfied in this case. □

Let x_{ij}, y_{ij} be commutative variables, and X, Y be $n\times k$ matrices with matrix elements x_{ij}, y_{ij}. Also, let ∂_X and ∂_Y be $n\times k$ matrices with the matrix elements $\frac{\partial}{\partial x_{i,j}}$ and $\frac{\partial}{\partial y_{i,j}}$. Let z be a variable commuting with y_{ij}. The following $2n\times 2k$, and $(n+k)\times(n+k)$ matrices, with elements in the ring of differential operators in the variables X, Y are easily seen to be Manin matrices:

$$\begin{pmatrix}X & \partial_Y\\ Y & \partial_X\end{pmatrix}, \qquad \text{and} \qquad \begin{pmatrix}z\,1_{k\times k} & (\partial_Y)^t\\ Y & \partial_z\,1_{n\times n}\end{pmatrix}. \tag{4}$$

2.1. The determinant

Definition 3. *Let M be a Manin matrix. Define the determinant of M by column expansion:*

$$\det M = \det^{\text{column}} M = \sum_{\sigma \in S_n} (-1)^{\sigma} \overset{\curvearrowright}{\prod_{i=1,\ldots,n}} M_{\sigma(i),i}, \tag{5}$$

where S_n is the group of permutations of n letters, and the symbol $\curvearrowright$ means that in the product $\prod_{i=1,\ldots,n} M_{\sigma(i),i}$ one writes first the elements from the first column, then from the second column and so on and so forth.

Lemma 2. *The determinant of a Manin matrix does not depend on the order of the columns in the column expansion, i.e.,*

$$\forall\, p \in S_n \det^{\text{column}} M = \sum_{\sigma \in S_n} (-1)^{\sigma} \overset{\curvearrowright}{\prod_{i=1,\ldots,n}} M_{\sigma(p(i)),p(i)}. \tag{6}$$

Proof. Since any permutation can be presented as a product of transpositions of neighbors $(i, i+1)$ it is enough to prove the proposition for such transpositions. But for them it follows from the equality of the commutators of cross elements (formula (1)). □

Example 1. For the case $n = 2$, we have

$$\det^{\text{col}} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \overset{\text{def}}{=} ad - cb \overset{\text{lemma}}{=} da - bc. \tag{7}$$

Indeed, this is a restatement of the second condition of definition 1.

2.1.1. Elementary properties. The following properties are simple consequences of the definition of Manin matrix.

(1) Any matrix with commuting elements is a Manin matrix.
(2) Any submatrix of a Manin matrix is again a Manin matrix.
(3) If A, B are Manin matrices and $\forall\, i, j, k, l : [A_{ij}, B_{kl}] = 0$, then $A + B$ is again a Manin matrix.
(4) If A is a Manin matrix, c is constant, then cA is a Manin matrix.
(5) If A is a Manin matrix, C is a constant matrix, then CA and AC are Manin matrices and $\det(CA) = \det(AC) = \det(C)\det(A)$.
(6) If A, B are Manin matrices and $\forall\, i, j, k, l : [A_{ij}, B_{kl}] = 0$, then AB is a Manin matrix and $\det(AB) = \det(A)\det(B)$.
(7) If A is a Manin matrix, then one can exchange the ith and the jth columns (rows); one can put ith column (row) on jth place (erasing jth column (row)); one can add new column (row) to matrix A which is equal to one of the columns (rows) of the matrix A; one can add the ith column (row) multiplied by any constant to the jth column (row); in all cases the resulting matrix will be again a Manin matrix.
(8) If A and simultaneously A^t are Manin matrices, then all elements $A_{i,j}$ commute with each other. (A q-analog of this lemma says that if A and simultaneously A^t are q-Manin, then A is a quantum matrix: '$R\overset{1}{A}\overset{2}{A} = \overset{2}{A}\overset{1}{A}R$' [Man87, Man88].)
(9) The exchange of two columns in a Manin matrix changes the sign of the determinant. If two columns or two rows in a Manin matrix M coincide, then $\det(M) = 0$.

Proof. The first assertion is obvious. For the second (in the case of equal ith and jth columns) use the column expansion of $\det(M)$, taking any permutation with the first two elements i and j. One sees that $\det(M)$ is a sum of elements of the form $(xy - yx)(z)$, where x, y belong to the same column. Using the column commutativity $[x, y] = 0$ we get the result. For the case of two coinciding rows the assertion follows at once, without the help of column commutativity. □

(10) Since any submatrix of a Manin matrix is a Manin matrix one has a natural definition of minors and again one can choose an arbitrary order of columns (rows) to define their determinant.

3. Lax matrices as Manin matrices

In the modern theory of integrable systems, 'Lax matrices' are also called 'Lax operators', 'transfer matrices' and 'monodromy matrices' for historical reasons. It is outside of the aim of the paper to make an attempt to provide a kind of formal definition of a Lax matrix for an integrable system. However, it is fair to say that there is a set of properties expected from a 'good' Lax matrix; let us now recall these properties relevant for our exposition.

Let M be a symplectic manifold and H_i be a set of Poisson-commuting functions defining an integrable system on it (for simplicity, let us consider the Hamiltonian flow X_1 of one of these functions, say H_1). A *Lax matrix* $L(z)$ for (M, H_i) is a matrix, whose matrix elements are functions on M, possibly depending[6] on a formal parameter z, satisfying the following two characteristic properties:

(1) The evolution of $L(z)$ along X_1 is of the form

$$\dot{L}(z) = [M_1(z), L(z)].$$

(2) *The characteristic polynomial* $\det(\lambda - L(z)) = \sum_{i,j} H_{i,j} z^j \lambda^i$ produces 'all Liouville' integrals of motion, i.e.

- $\forall\, i, j\, H_{i,j}$ Poisson commute among themselves and with the given functions H_i
$$\forall\, k, i, j \quad \{H_k, H_{i,j}\} = 0, \qquad \forall \quad i, j, k, l\{H_{i,j}, H_{k,l}\} = 0 \tag{8}$$
- All H_i can be expressed via $H_{k,l}$ and vice versa.

Poisson algebras of commutative functions on manifolds M are related to classical mechanics. As is well known, in quantum mechanics one is led to consider a family of non-commutative, but associative algebras $\widehat{Fun(M)}_\hbar$; $\hbar$ is a formal parameter in mathematics and Plank's constant in physics. For $\hbar = 0$ the algebra $\widehat{Fun(M)}_\hbar$ coincides with the commutative algebra $Fun(M)$ of functions on a manifold M, as well as Poisson brackets are related to commutators: $[f, g] = \hbar\{f, g\} mod(\hbar^2)$, for $f, g \in \widehat{Fun(M)}_\hbar$ (see, e.g., [K97]).

The standard example is the algebra of functions on $\mathbb{C}^{2n}$–$\mathbb{C}[p_i, q_i]$ with the Poisson bracket $\{p_i, q_j\} = \delta_{i,j}, \{p_i, p_j\} = \{q_i, q_j\} = 0$ whose quantization is the Heisenberg (Weyl) algebra generated by $\hat{p}_i, \hat{q}_i$ and the relations $[\hat{p}_i, \hat{q}_j] = \hbar\delta_{i,j}, [\hat{p}_i, \hat{p}_j] = [\hat{q}_i, \hat{q}_j] = 0$. (We will usually put $\hbar = 1$.)

Within this framework, on can look for a *quantum Lax matrix* for a quantum systems as a matrix satisfying

- $\hat{L}(z)$ is a matrix whose matrix elements are elements from $\widehat{Fun(M)}_\hbar$, usually depending on a formal parameter z (thus, a quantum Lax matrix is a matrix with *noncommutative* elements).

[6] Let us assume for simplicity of presentation that $L(z)$ is just a polynomial function of the formal parameter z.

- in the classical limit $\hbar \to 0$ one has $\hat{L}(z) \to L(z)^{\text{classical}}$.

It is quite natural to look, in the integrable case, for a kind of determinantal formula: '$\det(\hat{\lambda} - \hat{L}(\hat{z}))$' $= \sum_k \hat{\lambda}^k \hat{H}_k(\hat{z})$ to produce quantum integrals of motion: $[\hat{H}_k(z), \hat{H}_l(u)] = 0$, and, possibly, to satisfy other important properties (see [CT06]).

3.1. The Gaudin case

The algebra of symmetries of the gl_n-Gaudin [Ga76, Ga83] integrable system is the Lie algebra $gl[t]$. The Lax matrix for the Gaudin system is a convenient way to combine generators of $gl[t]$ (or its factor algebras) into one generating matrix-valued function (see formula (15*b*) below for $gl[t]$ itself). Quantum commuting Hamiltonians (quantum conservation laws) arise from the maximal commutative Bethe subalgebra in $U(gl_n[t])$. Explicit efficient expressions for generators were first obtained by D Talalaev. This subalgebra is actually an image of the center of $U_{c=\text{crit}}(gl_n[t, t^{-1}], c)$ under the natural projection $U_{c=\text{crit}}(gl_n[t, t^{-1}], c) \to U(gl_n[t])$ (see appendix A.1). This explains the mathematical meaning and the importance of the subject.

Definition 4. *Let R be an associative algebra over $\mathbb{C}$. Let us call a matrix $L(z)$ with elements in $R((z))$ (i.e. $L(z) \in Mat_n \otimes R \otimes \mathbb{C}((z))$)*[7] *a Lax matrix of gl_n-Gaudin type iff:*

$$[L_{ij}(u), L_{kl}(v)] = \frac{1}{u-v}(L_{il}(v)\delta_{jk} - L_{il}(u)\delta_{jk} - L_{kj}(v)\delta_{li} + L_{kj}(u)\delta_{li}). \tag{9}$$

More precisely this is 'rational' gl_n-Gaudin type; as is well known, there are trigonometric and elliptic versions as well as generalizations to semisimple Lie (super)-algebras.

We recall the matrix ('Leningrad's') notation: let $\Pi \in Mat_n \otimes Mat_n$ be the permutation matrix: $\Pi(a \otimes b) = b \otimes a$, and consider $L(z) \otimes 1, 1 \otimes L(u) \in Mat_n \otimes Mat_n \otimes R \otimes \mathbb{C}((z)) \otimes \mathbb{C}((u))$. Formula (9) can be compactly written as follows:

$$[L(z) \otimes 1, 1 \otimes L(u)] = \left[\frac{\Pi}{z-u}, L(z) \otimes 1 + 1 \otimes L(u)\right], \tag{10}$$

that is, a Lax matrix of gl_n-Gaudin type is a Lax matrix with a *linear* r-matrix structure (the r-matrix being $r = \frac{\Pi}{z-u}$).

Remark 3. From (9) it follows that

$$[L_{ij}(u), L_{kl}(u)] = -(\partial_u L_{il}(u)\delta_{jk} - \partial_u L_{kj}(u)\delta_{li}).$$

Here is our *first main observation*:

Proposition 2. *Consider $\partial_z \pm L(z) \in Mat_n \otimes R \otimes \mathbb{C}((z))[\partial_z]$, where $L(z)$ is a Lax matrix of the gl_n-Gaudin type above; then:*

$$(\partial_z - L(z)), \quad (\partial_z + L(z))^t \quad \textit{are Manin matrices}. \tag{11}$$

Proof. The proof is a straightforward computation. □

Example 2. Let us show this property in the example of 2×2 case:

$$\begin{aligned} &\partial_z - L(z) = \begin{pmatrix} \partial_z - L_{11}(z) & -L_{12}(z) \\ -L_{21}(z) & \partial_z - L_{22}(z) \end{pmatrix}, \\ &\text{column 1 commutativity:} \quad [\partial_z - L_{11}(z), -L_{21}(z)] = -L'_{21}(z) + [L_{11}(z), L_{21}(z)] = 0, \\ &\text{cross-term relation:} \quad [\partial_z - L_{11}(z), \partial_z - L_{22}(z)] = -L'_{22}(z) + L'_{11}(z) = [L_{21}(z), L_{12}(z)]. \end{aligned} \tag{12}$$

[7] Or $L(z) \in Mat_n \otimes R \otimes \mathbb{C}((z^{-1}))$; in many cases $L(z)$ is just a rational function of z.

The following well-known fact easily follows from 10:

Proposition 3. *Let $L(z)$ and $\tilde{L}(z)$ be Lax matrices of the Gaudin type with pairwise commuting elements then $L(z) + \tilde{L}(z)$ is again a Lax matrix of the Gaudin type.*

The classical counterpart of the commutation relations (9)—namely, with Poisson brackets replacing commutators and of the corresponding Lax matrices are associated with a large number of integrable systems, from the Neumann–Rosochatius systems to the Nahm's monopole equations (see, e.g., [BBT03] for a recent update of the list). In this paper, we shall concentrate on the following few examples of Gaudin-type Lax matrices.

In this paper, we shall concentrate on the following few examples of Gaudin-type Lax matrices.

The simplest example. Let K be an arbitrary constant matrix, and $n, k \in \mathbb{N}$, and $z_1, \dots, z_k$ be arbitrary points in the complex plane. Consider

$$\begin{aligned} L(z) &= K + \sum_{i=1,\dots,k} \frac{1}{z - z_i} \begin{pmatrix} \hat{q}_{1,i} \\ \cdots \\ \hat{q}_{n,i} \end{pmatrix} (\hat{p}_{1,i} \quad \cdots \quad \hat{p}_{n,i}) \\ &= K + \hat{Q}\,\mathrm{diag}\left(\frac{1}{(z - z_1)}, \dots, \frac{1}{(z - z_k)}\right)\hat{P}^t \end{aligned} \tag{13}$$

where $\hat{p}_{i,j}, \hat{q}_{i,j}, i = 1, \dots, n; j = 1, \dots, k$, are the standard generators of the standard Heisenberg algebra $[\hat{p}_{i,j}, \hat{q}_{k,l}] = \delta_{i,k}\delta_{j,l}$, $[\hat{p}_{i,j}, \hat{p}_{k,l}] = [\hat{q}_{i,j}, \hat{q}_{k,l}] = 0$. Also, $\hat{Q}, \hat{P}$ are $n \times k$-rectangular matrices with elements $\hat{Q}_{i,j} = \hat{q}_{i,k}$, $\hat{P}_{i,j} = \hat{p}_{i,j}$.

One can see that this Lax matrix satisfies relations 10 (9), and so by proposition 2 above $(\partial_z - L(z))$, $(\partial_z + L(z))^t$ are Manin matrices.

The standard example Consider $gl_n \oplus \cdots \oplus gl_n$ and denote by e^i_{kl} the standard basis element from the ith copy of the direct sum $gl_n \oplus \cdots \oplus gl_n$. The standard Lax matrix for the Gaudin system is:

$$L_{gl_n\text{–Gaudin standard}}(z) = \sum_{i=1,\dots,k} \frac{1}{z - z_i} \begin{pmatrix} e^i_{1,1} & \cdots & e^i_{1,n} \\ \cdots & \cdots & \cdots \\ e^i_{n,1} & \cdots & e^i_{n,n} \end{pmatrix} \tag{14}$$

$L_{gl_n\text{–Gaudin standard}}(z) \in Mat_n \otimes U(gl_n \oplus \cdots \oplus gl_n) \otimes \mathbb{C}(z)$.

One can see that this Lax matrix satisfies relations 10 (9), and so by proposition 2 above $\partial_z - L_{gl_n\text{–Gaudin standard}}(z)$ and $(\partial_z + L_{gl_n\text{–Gaudin standard}}(z))^t$ are Manin matrices.

The gl_n and $gl_n[t]$ examples. Consider the Lie algebra gl_n, with e_{ij} its standard linear basis; consider the polynomial Lie algebra $gl_n[t]/t^k$, $k = 1, \dots, \infty$. The Lax matrices for gl_n and $gl_n[t]/t^k$ are the following:

$$\begin{aligned} L_{gl_n}(z) &= \frac{1}{z} \begin{pmatrix} e_{1,1} & \cdots & e_{1,n} \\ \cdots & \cdots & \cdots \\ e_{n,1} & \cdots & e_{n,n} \end{pmatrix} \\ L_{gl_n[t]}(z) &= \sum_{i=1,\dots,k} \frac{1}{z^i} \begin{pmatrix} e_{1,1}t^{i-1} & \cdots & e_{1,n}t^{i-1} \\ \cdots & \cdots & \cdots \\ e_{n,1}t^{i-1} & \cdots & e_{n,n}t^{i-1} \end{pmatrix}. \end{aligned} \tag{15}$$

$L_{gl_n}(z) \in Mat_n \otimes U(gl_n) \otimes \mathbb{C}(z)$; $L_{gl_n[t]}(z) \in Mat_n \otimes U(gl_n[t]) \otimes \mathbb{C}((z^{-1}))$, where $U(g)$ is the universal enveloping algebra of g. Note, $zL_{gl_n}(z)$ coincide with expression (12) in section 2.1 [Kir00].

One can see that these Lax matrices satisfy relations 10 (9), and so by proposition 2 above $\partial_z - L_{gl_n}(z)$, $\partial_z - L_{gl_n[t]}(z)$, $(\partial_z + L_{gl_n}(z))^t$ and $(\partial_z + L_{gl_n[t]}(z))^t$ are Manin matrices.

3.2. The Yangian (Heisenberg chain) case

The algebra of symmetries of the Heisenberg XXX spin chain, the quantum Toda system and several other integrable systems is a Hopf algebra called Yangian. It was implicitly defined in the works of Faddeev's school; a concise mathematical treatment and deep results were given in [Dr85] (see also the excellent more recent surveys [Mol02, MNO94]). It is a deformation of the universal enveloping algebra of $gl[t]$. A Lax matrix of Yangian type is a convenient way to combine generators of the Yangian (or its factor algebras) into one generating matrix-valued function. Following standard notation we will write $T(z)$ ('transfer matrix') instead of $L(z)$ in the case of the Yangian-type Lax matrices.

Definition 5. *Let $\mathcal{R}$ be an associative algebra over $\mathbb{C}$. Let us call a matrix $T(z)$ with elements in $\mathcal{R}((z))$ (i.e. $T(z) \in Mat_n \otimes \mathcal{R} \otimes \mathbb{C}((z))$)[8] a Lax matrix of Yangian type iff:*

$$[T_{ij}(u), T_{kl}(v)] = \frac{1}{u-v}(T_{kj}(u)T_{il}(v) - T_{kj}(v)T_{il}(u)). \tag{16}$$

(See [Mol02], page 4, formula (2.3). More precisely this is the case of the gl_n-Yangian, there are generalizations to the semisimple Lie (twisted)-(super)-algebras.)

In matrix (Leningrad's) notation we have, with $\Pi \in Mat_n \otimes Mat_n$ the permutation matrix: $\Pi(a\otimes b) = b\otimes a$, and $T(z)\otimes 1$, $1\otimes T(u) \in Mat_n\otimes Mat_n\otimes\mathcal{R}\otimes\mathbb{C}((z))\otimes\mathbb{C}((u))$, $R(z-u) = \left(1\otimes 1 - \frac{\Pi}{z-u}\right)$ that formula (9) can be written as follows, as a *quadratic* R-matrix relation: (see [Mol02], page 6, proposition 2.3, formula (2.14):

$$\left(1\otimes 1 - \frac{\Pi}{z-u}\right)(T(z)\otimes 1)(1\otimes T(u)) = (1\otimes T(u))(T(z)\otimes 1)\left(1\otimes 1 - \frac{\Pi}{z-u},\right) \tag{17}$$

or in shortly: $R(z-u)\overset{1}{T}(z)\overset{2}{T}(u) = \overset{2}{T}(u)\overset{1}{T}(z)R(z-u).$ (18)

Our *second main observation* is:

Proposition 4. *If $T(z)$ is a Lax matrix of the Yangian type then $e^{-\partial_z}T(z)$, $(e^{\partial_z}T(z))^t$ are Manin matrices.*

Here $e^{-\partial_z}T(z) \in Mat_n \otimes Y(gl_n) \otimes \mathbb{C}[[1/z, e^{-\partial_z}]]$ (see definition 5).

Proof. As in the Gaudin-type case, it follows from a straightforward computation. □

The following well-known fact easily follows from relation 17:

Proposition 5. *Let $T(z)$ and $\tilde{T}(z)$ be Lax matrices of Yangian type with pairwise commuting elements; then the product $T(z)\tilde{T}(z)$ is again a Lax matrix of Yangian type.*

Let us herewith list a couple of remarkable examples of Yangian-type Lax matrices.

The Toda system. Consider the Heisenberg algebra generated by $\hat{p}_i, \hat{q}_i, i = 1,\ldots,n$, and relations $[\hat{p}_i, \hat{q}_j] = \delta_{i,j}$, $[\hat{p}_i, \hat{p}_j] = [\hat{q}_i, \hat{q}_j] = 0$. Define

$$T_{\text{Toda}}(z) = \prod_{i=1,\ldots,n} \begin{pmatrix} z - \hat{p}_i & e^{-\hat{q}_i} \\ -e^{\hat{q}_i} & 0 \end{pmatrix}. \tag{19}$$

[8] Or $T(z) \in Mat_n \otimes \mathcal{R} \otimes \mathbb{C}((z^{-1}))$; in many cases $T(z)$ is just a rational function of z.

One can see that this Lax matrix satisfies relations 17 (16), and so by proposition 4 $e^{-\partial_z} T_{\mathrm{Toda}}(z)$ is a Manin matrix. One can easily see that, with the identification $q_{n+1} = q_1$, the coefficient C_{n-1} of z^{n-2} of Tr $T_{\mathrm{Toda}}(z)$ equals $\sum_{k<l\leqslant n} \hat{p}_k \hat{p}_l - \sum_{i=1,\dots,n} e^{\hat{q}_i - \hat{q}_{i+1}}$, and the coefficient C_{n-1} of z^{n-1} is $-\sum_{i=1}^{n} p_i$. Thus $\frac{1}{2} C_{n-1}^2 - C_{n-2}$ is the physical Hamiltonian (i.e., the energy) of the periodic Toda chain.

The Heisenberg's XXX system. Consider the Heisenberg algebra generated by $[\hat{p}_{i,j}, \hat{q}_{k,l}] = \delta_{i,k}\delta_{j,l}$, $[\hat{p}_{i,j}, \hat{p}_{k,l}] = [\hat{q}_{i,j}, \hat{q}_{k,l}] = 0$; the quantum Lax matrix for the simplest case of the gl_n-Heisenberg spin-chain integrable systems can be given as follows:

$$T_{XXX\text{'simplest'}}(z) = \prod_{i=1,\dots,k} \left(1_{n\times n} + \frac{1}{z - z_i} \begin{pmatrix} \hat{q}_{1,i} \\ \cdots \\ \hat{q}_{n,i} \end{pmatrix} \begin{pmatrix} \hat{p}_{1,i} & \cdots & \hat{p}_{n,i} \end{pmatrix} \right). \tag{20}$$

One can see that this Lax matrix satisfies relations 17 (16), and so by proposition 4 $e^{-\partial_z} T_{XXX}(z)$ is a Manin matrix.

Consider $gl_n \oplus \cdots \oplus gl_n$ and denote by e_{kl}^i the standard basis element of the ith copy of the direct sum $gl_n \oplus \cdots \oplus gl_n$. The standard Lax matrix for the $GL(N)$–Heisenberg's XXX system is:

$$T_{gl_n-XXX-\mathrm{standard}}(z) = \prod_{i=1,\dots,k} \left(1_{n\times n} + \frac{1}{z - z_i} \begin{pmatrix} e_{1,1}^i & \cdots & e_{1,n}^i \\ \cdots & \cdots & \cdots \\ e_{n,1}^i & \cdots & e_{n,n}^i \end{pmatrix} \right) \tag{21}$$

$T_{gl_n-XXX-\mathrm{standard}}(z) \in Mat_n \otimes U(gl_n \oplus \cdots \oplus gl_n) \otimes \mathbb{C}(z)$.

One can see that this Lax matrix satisfies relations 17 (16), and so by proposition 4 $e^{-\partial_z} T_{gl_n-XXX-\mathrm{standard}}(z)$ is a Manin matrix.

4. Algebraic properties of Manin matrices and their applications

In this section we will derive a few less elementary properties of the Manin matrix, and give applications thereof to integrable systems.

4.1. Cramer's formula

Proposition 6 [Man88]. *Let M be a Manin matrix and denote by M^{adj} the adjoint matrix defined in the standard way (i.e. $M_{kl}^{adj} = (-1)^{k+l} \det^{\mathrm{column}}(\widehat{M}_{lk})$ where $\widehat{M}_{lk}$ is the $(n-1) \times (n-1)$ submatrix of M obtained removing the lth row and the kth column. Then the same formula as in the commutative case holds true, that is,*

$$M^{adj} M = \det^{\mathrm{column}}(M) Id. \tag{22}$$

If M^t is a Manin matrix, then M^{adj} is defined by row determinants and $M M^{adj} = \det^{\mathrm{column}}(M^t) Id = \det^{\mathrm{row}}(M) Id$.

Example 3. In the 2×2 case we have:

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} da - bc & db - bd \\ -ca + ac & -cb + ad \end{pmatrix} = \begin{pmatrix} ad - cb & 0 \\ 0 & ad - cb \end{pmatrix}, \tag{23}$$

where the characteristic commutation relations of a Manin matrix have been taken into account.

Proof. One can see that the equality: $\forall\, i : (M^{adj} M)_{i,i} = \det^{\mathrm{col}}(M)$, follows from the fact that $\det^{\mathrm{col}}(M)$ does not depend on the order of the column expansion of the determinant. This independence was proved above (lemma 2). Let us introduce a matrix $\tilde{M}$ as follows. Take the matrix M and set the ith column equal to the jth column; denote the resulting matrix by $\tilde{M}$. Note that $\det^{\text{col}}(\tilde{M}) = 0$ precisely gives $(M^{adj}M)_{i,j} = 0$ for $i \neq j$. To prove that $\det^{\text{col}}(\tilde{M}) = 0$ we argue as follows. Clearly $\tilde{M}$ is a Manin matrix. Lemma 2 allows us to calculate the determinant taking the elements first from the ith column, then the jth, then other elements from the other columns. Now it is quite clear that $\det^{\text{col}}(\tilde{M}) = 0$, since it is the sum of the elements of the form $(xy - yx)(z) = 0$, where x, y are the elements from the ith and jth of $\tilde{M}$, so from the jth column of M. By Manin's property elements from the same column of a matrix M commute, so $xy - yx = 0$, so $\det^{\text{col}}(\tilde{M}) = 0$. □

Remark 4. The only difference with the commutative case is that, in the equality (22) the order of the products of M^{adj} and M has to be kept in mind.

4.1.1. Application to the Knizhnik–Zamolodchikov equation. We can give a very simple proof of the formula relating the solutions of KZ with coupling constant $\kappa = 1$ to the solutions of the equation defined by Talalaev's formula:

$$\det(\partial_z - L(z))Q(z) = 0.$$

This result was first obtained in [CT04] in a more complicated manner (see also [CT06b]).

As is well known, The KZ equation is actually a system of equations, but due to the symmetry it is enough to consider only one equation, say the first. In general, the standard KZ equation [KZ84] for gl_n, and the particular choice of the first representation space to be $\mathbb{C}^n$, is given by:

$$\left(\partial_z - \kappa \sum_{i=1,\dots,k} \frac{\sum_{ab} E_{ab} \otimes \pi_i\left(e^{(i)}_{ab}\right)}{z - z_i}\right) \Psi(z) = \pi(\partial_z - \kappa L_{\text{Gaudin}}(z))\Psi(z) = 0,$$
$$\Psi(z) = \begin{pmatrix} \Psi_1(z) \\ \dots \\ \Psi_n(z) \end{pmatrix} \tag{24}$$

where $(\pi, V_1 \otimes \cdots \otimes V_k)$ is a representation of $U(\mathfrak{gl}_n)^{\otimes k}$ and $\Psi(z)$ is a $\mathbb{C}^n \otimes V_1 \otimes \cdots \otimes V_k$-valued function, so that its components $\Psi_i(z)$ are $V_1 \otimes \cdots \otimes V_k$-valued functions. We see that $L_{\text{Gaudin}}(z) = \sum_{i=1,\dots,k} \frac{\sum_{ab} E_{ab} \otimes \pi_i(e^{(i)}_{ab})}{z - z_i}$ is the standard Lax matrix of the quantum Gaudin system (see formula (14)) considered in a representation π. We denote by $E_{ab} \in Mat_n$ and $e_{ab} \in gl_n \subset U(gl_n)$ the standard matrix units.

Proposition 7. *Let $\Psi(z)$ be a solution of the KZ equation with $\kappa = 1$ (24), then:*

$$\forall\, i = 1, \dots, n \quad \pi(\det(\partial_z - L_{\text{Gaudin}}(z)))\Psi_i(z) = 0. \tag{25}$$

Proof. The adjoint matrix $(\partial_z - L_{\text{Gaudin}}(z))^{adj}$ exists as we discussed above (here we use $\kappa = 1$), so:

$$\begin{aligned} &\pi(\partial_z - L_{\text{Gaudin}}(z))\Psi(z) = 0, \Rightarrow \pi((\partial_z - L_{\text{Gaudin}}(z))^{adj})\pi(\partial_z - L_{\text{Gaudin}}(z))\Psi(z) = 0, \\ &\text{hence} \quad \pi(\det(\partial_z - L_{\text{Gaudin}}(z)))Id \quad \Psi(z) = 0, \\ &\text{hence } \forall\, i = 1, \dots, n \quad \pi(\det(\partial_z - L_{\text{Gaudin}}(z)))\Psi_i(z) = 0. \end{aligned} \tag{26}$$

□

Remark 5. The equation $\det(\partial_z - L(z))Q(z) = 0$ should be seen as generalized Baxter's $T - Q$-equation (see [CT06]), it is well known that Baxter's equation plays a crucial role in the solution of quantum systems. In particular, it is important to establish that its solutions are rational functions (see [MTV05]). The above proposition relates this question to the rationality of KZ solutions which is a more natural question (see [Sa06]).

4.2. Inversion of Manin matrices

Theorem 1. *Let M be a Manin matrix, and assume that a two-sided inverse matrix M^{-1} exists (i.e. $M^{-1}M = MM^{-1} = 1$). Then M^{-1} is again a Manin matrix.*

Proof. The proof of the assertion consists of a small extension and rephrasing of the arguments in the proof of lemma 1 (page 5 of [BT02]).

We consider the Grassman algebra $\mathbb{C}[\psi_1, \ldots, \psi_n]$ (i.e. $\psi_i^2 = 0, \psi_i\psi_j = -\psi_j\psi_i$), with ψ_i commuting with $M_{p,q}$. Let us introduce new variables $\tilde{\psi}_i$ by

$$(\tilde{\psi}_1, \ldots, \tilde{\psi}_n) = (\psi_1, \ldots, \psi_n) \begin{pmatrix} M_{1,1} & \cdots & M_{1,n} \\ \cdots & & \\ M_{n,1} & \cdots & M_{n,n} \end{pmatrix}. \tag{27}$$

It is easy to see that the Manin relations $\left[M_{ij}^{-1}M_{kl}^{-1}\right] = \left[M_{kj}^{-1}, M_{il}^{-1}\right]$ we have to prove follow from the equality

$$\begin{aligned} M_{ij}^{-1}M_{kl}^{-1}\psi_1 \wedge \cdots \wedge \psi_n &= M_{kj}^{-1}M_{il}^{-1}\psi_1 \wedge \cdots \wedge \psi_n \\ &\quad + (\det(M))^{-1}\tilde{\psi}_1 \wedge \tilde{\psi}_2 \wedge \cdots \wedge \overset{\text{ithplace}}{\psi_j} \wedge \cdots \wedge \overset{\text{kthplace}}{\psi_l} \wedge \cdots \wedge \tilde{\psi}_n. \end{aligned} \tag{28}$$

(Here assume $i \neq k$; otherwise the desired result is a tautology.) Let us prove (28). By the definition of the $\tilde{\psi}_i$'s, $(\tilde{\psi}_1, \ldots, \tilde{\psi}_m) = (\psi_1, \ldots, \psi_n)M$, multiplying this relation by M^{-1} on the right we get

$$(\tilde{\psi}_1, \ldots, \tilde{\psi}_n)M^{-1} = (\psi_1, \ldots, \psi_n), \quad \Leftrightarrow \quad \sum_v \tilde{\psi}_v M_{vl}^{-1} = \psi_l.$$

We multiply this relation with the $n-1$-vector $\tilde{\psi}_1 \wedge \tilde{\psi}_2 \wedge \cdots \wedge \overset{i\text{th place}}{\psi_j} \wedge \cdots \wedge \overset{k\text{th place}}{empty} \wedge \cdots \wedge \tilde{\psi}_n$, and use the fact that $\forall\, m : \tilde{\psi}_m^2 = 0$, to get

$$\left(\tilde{\psi}_i M_{il}^{-1} + \tilde{\psi}_k M_{kl}^{-1} - \psi_l\right)\tilde{\psi}_1 \wedge \tilde{\psi}_2 \wedge \cdots \wedge \overset{i\text{th]place}}{\psi_j} \wedge \cdots \wedge \overset{k\text{th place}}{empty} \wedge \cdots \wedge \tilde{\psi}_n = 0.$$

Now, using: $\det(M)M_{ji}^{-1}\psi_1 \wedge \cdots \wedge \psi_n = \tilde{\psi}_1 \wedge \cdots \wedge \overset{j\text{th place}}{\psi_i} \wedge \cdots \wedge \tilde{\psi}_n$ we get the desired result. □

Remark 6. The paper [Ko07B] (see section 5, page 11, proposition 5.1) contains a somewhat weaker proposition for matrices of the form $1 - tM$, where t is a formal parameter. It states that for $(1 - tM)^{-1}$ the cross-commutation relations hold true. The 'column commutation' property is not considered there. The equality $\det^{\text{column}}((1-tM)^{-1}) = (\det^{\text{column}}(1-tM))^{-1}$ is contained in theorem 5.2, page 13 of [Ko07B]. The proofs presented there are based on deep combinatorial arguments.

4.2.1. Application to the Enriquez–Rubtsov–Babelon–Talon theorem. Let us show that the remarkable theorem [ER01] (theorem 1.1, page 2), [BT02] (theorem 2, page 4) about 'quantization' of separation relations—follows as a particular case from theorem 1.

Let us briefly recall the constructions of this theorem (following, for simplicity, [BT02]). Let $\{\alpha_i, \beta_i\}_{i=1,\ldots,g}$ be a set of quantum 'separated' variables, i.e. satisfying the commutation relations

$$[\alpha_i, \alpha_j] = 0, \quad [\beta_i, \beta_j] = 0, \quad [\alpha_i, \beta_j] = f(\alpha_i, \beta_i)\delta_{ij}, \quad i, j, = 1, \ldots, g.$$

satisfying a set of equations (i.e., quantum Jacobi separation relations) of the form

$$\sum_{j=1}^{g} B_j(\alpha_i, \beta_i) H_j + B_0(\alpha_i, \beta_i) = 0, \qquad i = 1, \ldots, g, \tag{29}$$

for a suitable set of quantum Hamiltonians $H_1, \ldots, H_n$. One assumes that some ordering in the expressions $B_a, a = 0, \ldots, g$, between α_i, β_i has been chosen, and that the operators H_i are, as is written above, on the right of the B_j. Also, the B_a are some functions (say, polynomials) of their arguments with $\mathbb{C}$-number coefficients. Then the statement is that the quantum operators $H_1, \ldots, H_g$ fulfilling (29) commute among themselves.

Our proof starts from the fact that one can see that equations (29) can be compactly written, in matrix form, as

$$B \cdot H = -V, \qquad \text{with} \quad B_{ij} = B_j(\alpha_i, \beta_i), \quad V_i = B_0(\alpha_i, \beta_i),$$

and thus (making contact with the formalism of [ER01]) one is led to consider the $g \times (g+1)$ matrix

$$A = \begin{pmatrix} V_1 & B_{1,1} & \cdots & B_{1,g} \\ \cdots & \cdots & \cdots & \cdots \\ V_g & B_{g,1} & \cdots & B_{g,g} \end{pmatrix}.$$

Thanks to the functional form of the B_{ij}'s and of the V_i's, this matrix is a Cartier–Foata matrix (i.e., elements from different rows commute among each other), and hence, *a fortiori* a Manin matrix.

Given such a $g \times (g+1)$ Cartier–Foata matrix A, we consider the following $(g+1) \times (g+1)$ Cartier–Foata (and hence, Manin) matrix:

$$\tilde{A} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ V_1 & B_{1,1} & \cdots & B_{1,g} \\ \cdots & \cdots & \cdots & \cdots \\ V_g & B_{g,1} & \cdots & B_{g,g} \end{pmatrix}.$$

Now it is obvious that the solutions H_i of equation (29) are the elements of the first column of the inverse of $\tilde{A}$, and namely

$$H_i = (\tilde{A}^{-1})_{i+1,1}, \qquad i = 1, \ldots, g.$$

Since $\tilde{A}$ is a Cartier–Foata, its inverse is Manin, and thus the commutation of the H_i's can be obtained from theorem 1.

Remark 7. For the sake of simplicity, we considered, as in [BT02], an arbitrary system for which a quantized spectral curve exists, and the commuting Hamiltonians can be written by the above formula in terms of the so-called separated variables (see appendix A.2). However, it should be noted that the theorem holds in a more general (in classical terms, Stäckel or Jacobi) setting, where the functional dependence of the matrix elements $B_{ij}(\alpha_i, \beta_i)$ and $V_i(\alpha_i, \beta_i)$ on their argument may depend on the index i.

4.3. Block matrices and Schur's complement

Theorem 2. *Consider a Manin matrix M of size n, and denote its block as follows:*

$$M = \begin{pmatrix} A_{k\times k} & B_{k\times n-k} \\ C_{n-k\times k} & D_{n-k\times n-k} \end{pmatrix}. \tag{30}$$

Assume that M, A, D are invertible i.e. $\exists M^{-1}, A^{-1}, D^{-1} : A^{-1}A = AA^{-1} = 1, D^{-1}D = DD^{-1} = 1, M^{-1}M = MM^{-1} = 1$. Then the same formulae as in the commutative case hold, *namely:*

$$\det^{\mathrm{col}}(M) = \det^{\mathrm{col}}(A)\det^{\mathrm{col}}(D - CA^{-1}B) = \det^{\mathrm{col}}(D)\det^{\mathrm{col}}(A - BD^{-1}C), \tag{31}$$

and, moreover, the Schur's complements: $D - CA^{-1}B$, and $A - BD^{-1}C$ are Manin matrices.

Sketch of the Proof. The full proof of this fact will be given in [CFRu]. Actually it is not difficult to see that $A - BD^{-1}C$, $D - CA^{-1}B$ are Manin matrices from theorem 1, so basically we need to prove the determinantal formula (31). To this end, one proves the □

Lemma 3. *Let M be a Manin matrix of size $n \times n$. Let X be a $k \times (n-k)$ matrix, for some k, with arbitrary matrix elements*[9]. *Then:*

$$\det^{\mathrm{column}}\left(M\begin{pmatrix} 1_{k\times k} & X_{k\times n-k} \\ 0_{n-k\times k} & 1_{n-k\times n-k}\end{pmatrix}\right) = \det^{\mathrm{column}} M. \tag{32}$$

Now, the desired result follows now from the equality:

$$\begin{pmatrix} A & B \\ C & D\end{pmatrix}\begin{pmatrix} 1 & -A^{-1}B \\ 0 & 1\end{pmatrix} = \begin{pmatrix} A & 0 \\ C & D - CA^{-1}B\end{pmatrix}. \tag{33}$$

Remark 8. (1) Alternatively, one can prove (31) using the Jacobi-type theorem by I Gelfand, V Retakh (see, e.g., [GR97], theorem 1.3.3, page 8) with similar arguments to those used by D Krob, B Leclerc [KL94] (theorem 3.2, page 17). (2) The Jacobi ratio theorem was proved for Manin matrices of the form $1 - tM$, t is a formal parameter, in the remarkable paper [Ko07B] (see theorem 5.2, page 13). It is actually equivalent to our formula (31) for matrices of such a form. As in the case of the matrix inversion formula, column commutativity is not considered, and the proofs therein contained are based on combinatorial properties.

4.3.1. Application. MTV–Capelli identity for $gl[t]$ (for the Gaudin system). In [MTV06] a remarkable generalization of the Capelli identity has been found. Using the beautiful insights contained therein, we remark that it follows from theorem 2.

Consider $\mathbb{C}[p_{i,j}, q_{i,j}]$, $i = 1, \ldots, n$; $j = 1, \ldots, k$, endowed with the standard Poisson bracket:

$\{p_{i,j}, q_{k,l}\} = \delta_{i,k}\delta_{j,l}, \{p_{i,j}, p_{k,l}\} = \{q_{i,j}, q_{k,l}\} = 0$. Consider its quantization i.e. the standard Heisenberg (Weyl) associative algebra generated by $\hat{p}_{i,j}, \hat{q}_{i,j}, i = 1, \ldots, n$; $j = 1, \ldots, k$, with the relations $[\hat{p}_{i,j}, \hat{q}_{k,l}] = \delta_{i,k}\delta_{j,l}, [\hat{p}_{i,j}, \hat{p}_{k,l}] = [\hat{q}_{i,j}, \hat{q}_{k,l}] = 0$. Define $n \times k$-rectangular matrices $Q_{\mathrm{classical}}, P_{\mathrm{classical}}, \hat{Q}, \hat{P}$ as follows with: $Q_{i,j} = q_{i,j}$, $\hat{Q}_{i,j} = \hat{q}_{i,j}$, $P_{i,j} = p_{i,j}$, $\hat{P}_{i,j} = \hat{p}_{i,j}$. Let K_1, K_2 be $n \times n$, $k \times k$ matrices with elements in $\mathbb{C}$.

Let us introduce the following notations:

$$L^{\mathrm{quantum}}(z) = K_1 + \hat{Q}(z - K_2)^{-1}\hat{P}^t, \qquad L^{\mathrm{classical}}(z) = K_1 + Q_{\mathrm{classical}}(z - K_2)^{-1}P^t_{\mathrm{classical}} \tag{34}$$

These are Lax matrices of Gaudin type i.e. they satisfy the commutation relation (9).

Let us reformulate the result of [MTV06] in the following simple form:

Proposition 8. *Capelli identity for the $gl[t]$ (Gaudin's system)*

$$Wick(\det(\lambda - L^{\mathrm{classical}}(z))) = \det(\partial_z - L^{\mathrm{quantum}}(z)). \tag{35}$$

[9] No conditions of commutativity at all are required.

Here we denote by Wick the linear map: $\mathbb{C}[\lambda, p_{i,j}, q_{i,j}](z) \to \mathbb{C}[\partial_z, \hat{p}_{i,j}, \hat{q}_{i,j}](z)$*, defined as:*

$$Wick(f(z)\lambda^a \prod_{ij} q^{c_{ij}} \prod_{ij} p_{ij}^{b_{ij}}) = f(z)\partial_z^a \prod_{ij} \hat{q}_{ij}^{b_{ij}} \prod_{ij} \hat{p}_{ij}^{b_{ij}} \tag{36}$$

i.e. that, from a commutative monomial, makes a noncommutative monomial according to the rule that any $\hat{q}$ *placed on the left of any* $\hat{p}$*, and the same for* z *and* ∂_z*. It is sometimes called 'Wick or normal ordering' in physics.*

Proof. Following [MTV06] we consider the following block matrix:

$$MTV = \begin{pmatrix} z - K_2 & \hat{P}^t \\ \hat{Q} & \partial_z - K_1 \end{pmatrix}. \tag{37}$$

It is easy to see that MTV is a Manin matrix (indeed it is strictly related to the Manin matrices briefly introduced in (4)); also, as was observed by Mukhin, Tarasov and Varchenko, the Lax matrix of the form (34) appears as the Schur's complement '$D - CA^{-1}B$' of the matrix MTV:

$$\partial_z - K_1 - \hat{Q}(z - K_2)^{-1}\hat{P}^t = \partial_z - L^{\text{quantum}}(z).$$

By theorem 2 we get that

$$\det{}^{\text{column}}(MTV) = \det(z - K_2)\det{}^{\text{column}}(\partial_z - K_2 - \hat{Q}(z - K_2)^{-1}\hat{P}^t). \tag{38}$$

This is a form of the Capelli identity as explained in [MTV06].

In order to arrive at formula (35), we just remark that in $\det^{\text{column}}(MTV)$ all variables $z, \hat{q}_{ij}$ stand on the left of the variables $\partial_z, \hat{p}_{ij}$. This is due to the column expansion of the determinant, where first appears the first column, then the second, and so on and so forth. Actually, the operators $z, \hat{q}_{ij}$ stand in the first nth columns of the MTV matrix, while the operators $\partial_z, \hat{p}_{ij}$ are in the m rightmost columns. So dividing (38) by $\det(z - K_2)$ we obtain (35). □

5. Spectral properties of Manin matrices and applications

5.1. The Cayley–Hamilton theorem

The Cayley–Hamilton theorem (i.e, that any ordinary matrix satisfies its characteristic polynomial) can be considered as one of the basic results in linear algebra. The same holds—with a suitable proviso in mind—for Manin matrices.

Theorem 3. *Let* M *be a* $n \times n$ *Manin matrix. Consider its characteristic polynomial and the coefficients* h_i *of its expansion in powers of* t*:* $\det^{\text{column}}(t - M) = \sum_{i=0,\ldots,n} h_i t^i$*; then*

$$\sum_{i=0,\ldots,n} h_i M^i = 0, \qquad \text{i.e.} \quad \det{}^{\text{column}}(t - M)|_{t=M}^{\text{right substitute}} = 0. \tag{39}$$

If M^t *is a Manin matrix, then one should use left substitution and row determinant:* $\det^{\text{row}}(t - M)|_{t=M}^{\text{left substitute}} = 0.$

Proof. It was proved in proposition 6 that there exists an adjoint matrix $(M - t\,Id)^{adj}$, such that

$$(M - t\,Id)^{adj}(M - t\,Id) = \det(M - t\,Id)Id. \tag{40}$$

The standard idea of proof is very simple: we want to substitute M at the place of t; the LHS of this equality vanishes manifestly, hence we obtain the desired equality $\det(M - t\,Id)|_{t=M} = 0$.

The only issue we need to clarify is *how* to substitute M into the equation and why the substitution preserves the equality.

Let us denote by $Adj_k(M)$ the matrices defined by: $\sum_{k=0,\ldots,n-1} Adj_k(M)t^k = (M - t\,Id)^{adj}$. The above equality is an equality of polynomials in the variable t:

$$\left(\sum_k Adj_k(M)t^k\right)(M - t\,Id) = \sum_k Adj_k(M)Mt^k - \sum_k Adj_k(M)t^{k+1} \quad (41)$$

$$= \det(M - tId) = \sum_k h_k t^k. \quad (42)$$

This means that the coefficients of t^i of both sides of the relation coincide. Hence we can substitute $t = M$ in the equality, substituting 'from the right':

$$\sum_k Adj_k M M^k - \sum_k Adj_k M^{k+1} = \sum_k h_k t^k\big|_{t=M}. \quad (43)$$

The left-hand side is manifestly zero, so we obtain the desired equality: $\sum_{k=0,\ldots,n} h_k M^k = 0$. □

Remark 9. See [Mol02] (section 4.2), [GIOPS05] and references therein for other CH-theorems.

5.1.1. Example. The Cayley–Hamilton theorem for the Yangian. Let us consider the Lax matrix $T_{gl_n-\mathrm{Yangian}}(z)$ (or $T(z)$ for brevity) of Yangian type (see section 3.2). The matrix $\mathrm{e}^{-\partial_z}T(z)$ is a Manin matrix. Let us derive a Cayley–Hamilton identity for $T(z)$ from the one for the Manin matrix $\mathrm{e}^{-\partial_z}T(z)$.

Definition 6. *Let us define the 'quantum powers' $T^{[p]}(z)$ for the Yangian-type Lax matrix as follows:*

$$T^{[p]}(z) \stackrel{\mathrm{def}}{=} T(z+p-1)\cdot T(z+p-2)\cdot\cdots\cdot T(z) = \mathrm{e}^{p\partial_z}(\mathrm{e}^{-\partial_z}T(z))^p. \quad (44)$$

Denote by $QH_i(z)$[10] coefficients of the expansion of the 'ordered' characteristic polynomial in powers of e^{∂_z}:

$$\det{}^{\mathrm{column}}(t - \mathrm{e}^{-\partial_z}T(z)) = \sum_{k=0,\ldots,n} t^k QH_k(z)\,\mathrm{e}^{(k-n)\partial_z}, \quad (45)$$

Note that the operator ∂_z does not enter into the expressions $QH_i(z)$. Explicit formulae for $QH_i(z)$ are obviously the following:

$$QH_0(z) = (-1)^n qdet(T(z-1)) \qquad QH_{n-i}(z) = (-1)^{(n-i)} \sum_{j_1<\cdots<j_i} qdet(T(z-1)_{j_1,\ldots,j_i}).$$

In words, the $QH_i(z)$'s are the sums of principal q-minors of size i (in complete analogy with the commutative case, modulo substitution of minors by q-minors). Recall that $qdet(M(z))$ is defined by the formula $qdet(M(z)) = \sum_{\sigma\in S_n}\prod_k M_{\sigma(k),k}(z-k+1)$ (see, e.g., formula (2.24), page 10, [Mol02]).

[10] we slightly changed the definition of $QH_i(z)$ comparing with the version 1, and corrected the misprint in formula (45).

Proposition 9. *The following is an analogue of the Cayley–Hamilton theorem for Yangian-type Lax matrices:*

$$\sum_{k=0,\dots,n} QH_k(z+n)T^{[k]}(z)=0. \tag{46}$$

Proof. The proof is very simple: one uses the Cayley–Hamilton theorem 3 for the Manin matrix $e^{-\partial_z}T(z)$, and than excludes powers of e^{∂_z} from the identity[11]. We can proceed as follows. By theorem 3 it holds true:

$$\sum_{k=0,\dots,n} h_k(z,\partial_z)(e^{-\partial_z}T(z))^k=0,\quad\Rightarrow\quad \sum_{k=0,\dots,n} h_k(z,\partial_z)\,e^{-k\partial_z}T^{[k]}(z)=0, \tag{47}$$

where $h_k(z,\partial_z)$ are defined as $\det^{\mathrm{column}}(t-e^{-\partial_z}T(z))=\sum_{i=0,\dots,n} h_i(z,\partial_z)t^i$. By the definition of $QH_i(z)$ (formula (44)) one has: $h_i(z,\partial_z)=QH_i(z)\,e^{(i-n)\partial_z}$, substituting this into 47, one obtains:

$$0=\sum_{k=0,\dots,n} h_k(z,\partial_z)\,e^{-k\partial_z}T^{[k]}(z)=\sum_{k=0,\dots,n} QH_k(z)\,e^{(k-n)\partial_z}\,e^{-k\partial_z}T^{[k]}(z) \tag{48}$$

$$=\sum_{k=0,\dots,n} QH_k(z)\,e^{-n\partial_z}T^{[k]}(z)=e^{-n\partial_z}\sum_{k=0,\dots,n} QH_k(z+n)T^{[k]}(z), \tag{49}$$

So we conclude that $\sum_{k=0,\dots,n} QH_k(z+n)T^{[k]}(z)=0$. □

Remark 10. The quantities $QH_k(z)$ were defined in [KS81] (formula (5.6), page 114) by means of different arguments. The idea of obtaining them as coefficients of a suitable characteristic polynomial is due to D Talalaev [Ta04].

5.2. The Newton identities and applications

As is well known, the Newton identities are identities between power sums symmetric functions and elementary symmetric functions. They can be rephrased as relations between $\mathrm{Tr}\,M^k$ and coefficients of $\det(t+M)$ for matrices with commutative entries. We will show that the same identities hold true for Manin matrices and present applications.

Theorem 4. *Let M be a Manin matrix, and denote by h_k the coefficients of the expansion in t of the determinant of $t-M$, i.e., $\det^{\mathrm{column}}(t-M)=h_kt^k$; conventionally, let $h_k=0$ for $k<0$. Then the following identity holds:*

$$\partial_t \det^{\mathrm{column}}(t-M)=\frac{1}{t}(\det^{\mathrm{column}}(t-M))\sum_{k=0,\dots,\infty}\mathrm{Tr}(M/t)^k, \tag{50}$$

$$\Leftrightarrow \forall\, k: -\infty<k\leqslant n\quad \textit{it holds:}\quad kh_k=\sum_{i:\max(0,-k)\leqslant i\leqslant n-k} h_{k+i}\,\mathrm{Tr}(M)^i, \tag{51}$$

where n is the size of the matrix M. If M^t is a Manin matrix, then $\partial_t\det^{\mathrm{row}}(t-M)=(\det^{\mathrm{row}}(t-M))\left(1/t\sum_{k=0,\dots,\infty}\mathrm{Tr}(M/t)^k\right)$.

So these identities are identical to those of the commutative case, provided one pays attention to the order of terms: $\tilde{h}_i\,\mathrm{Tr}\,M^p$ if M is a Manin matrix ($\mathrm{Tr}\,M^p\tilde{h}_i$ if M^t is a Manin matrix). Obviously enough, this difference is due to the absence of the commutativity.

[11] One can easily exclude ∂_z in the Yangian case, but it is not clear at the moment how to do it in the Gaudin case. Nevertheless, a similar identity can be proved for the Gaudin case also: [CT06b] (by different methods).

Sketch of the proof. First observe the following simple property, whose proof is straightforward[12]:

$$\mathrm{Tr}(t+M)^{adj} = \partial_t \det(t+M).$$

So we consider

$$1/t \sum_{k=0,\ldots,\infty} \mathrm{Tr}((-M/t)^k) = \mathrm{Tr}\,\frac{1}{t+M} = \mathrm{Tr}((\det{}^{\mathrm{column}}(t+M))^{-1}(t+M)^{adj})$$
$$= (\det{}^{\mathrm{column}}(t+M))^{-1}\,\mathrm{Tr}(t+M)^{adj} = (\det{}^{\mathrm{column}}(t+M))^{-1}\partial_t \det{}^{\mathrm{column}}(t+M). \tag{52}$$

Substituting $-M$ instead of M one obtains the result. The case where M^t is a Manin matrix is similar. □

Remark 11. While working at the proof of this proposition we were informed by P. Pyatov (unpublished, based on Gurevich, Isaev, Ogievetsky, Pyatov, Saponov papers [GIOPS95, GIOPS05]) that the Newton identities hold true as well as their q-analogs. Our proof is different and very simple, but it is a challenge how to generalize such an argument to the case $q \neq 1$.

5.2.1. Quantum powers, $ZU_{\mathrm{crit}}(gl_n[t,t^{-1}],c)$, and new quantum Gaudin integrals. We recall from [CT06b] a possible definition of 'quantum powers' of the Lax matrix of the Gaudin model and apply the quantum Newton relations above to prove the commutativity of their traces. Hence we obtain new explicit generators in the center of $U_{c=\mathrm{crit}}(gl_n[t,t^{-1}],c)$ and in the commutative Bethe subalgebra in $U(gl_n[t]))$ (see appendix A.1) for a reminder); in a physicist's language we give explicit formulae for quantum conservation laws for the quantum Gaudin system. They are quantum analogs of $\mathrm{Tr}(L^{\mathrm{classical}}(z))^k$, and the result below exhibits the appropriate corrections for traces of powers which preserve commutativity.

Definition 7. *The quantum powers of Gaudin-type Lax matrices (definition 4, formula (9)) are defined inductively as follows:*

$$L^{[0]}(z) = Id, \qquad L^{[i]}(z) = L^{[i-1]}(z)L(z) + (L^{[i-1]}(z))'. \tag{53}$$

Here $(L^{[i-1]}(z))'$ is the derivative with respect to z of $(L^{[i-1]}(z))$. $L^{[i]}(z)$ are noncommutative analogues of the Faà di Bruno polynomials ([Di03] section 6A, page 111). (Remark that in the commutative case $L(z)$ and $L(z)'$ commute.) The binomial-type formula below is due to D Talalaev:

$$(\partial_z - L(z))^i = \sum_{p=0,\ldots,i} (-1)^p \binom{i}{p} \partial_z^{(i-p)} L^{[p]}(z). \tag{54}$$

Theorem 5. *Consider the quantum Hamiltonians $QH_i(z)$ defined by:* $\det(\partial_z - L(z)) = \sum_{i=0,\ldots,n} QH_i(z)\partial_z^i$. *Then:*

$$\forall\, k,l = 1,\ldots,n \text{ and } u,v \in \mathbb{C} \qquad [QH_k(u), \mathrm{Tr}\,L^{[l]}(v)] = 0, \quad [\mathrm{Tr}\,L^{[k]}(u), \mathrm{Tr}\,L^{[l]}(v)] = 0. \tag{55}$$

[12] Actually, the equality holds for any matrix M, provided one consistently defines the determinant and the adjoint matrix.

Sketch of the proof. The proof follows immediately from the Newton identities above, theorem of [Ta04] (see formula (2)) on the commutativity of coefficients of $\det(\partial_z - L(z)) = \sum_i QH_i(z)\partial_z^i$, and the binomial formula above. □

Corollary 1. *If $L(z) = L_{gl_n[t]}(z)$ is given by formula (lax-gltb), then $\mathrm{Tr}\, L^{[k]}_{gl_n[t]}(z)$ are generating functions (in the variable z) for the elements in a commutative Bethe subalgebra in $U(gl_n[t])$. Such elements for $k = 1, \ldots, n$ algebraically generate the Bethe subalgebra.*

Corollary 2. *If $L_{\mathrm{full}}(z)$ is given by formula (A.1), then $: \mathrm{Tr}\, L^{[k]}_{\mathrm{full}}(z) :$ are generating functions (in the variable z) for the elements in the center of $U_{c=\mathrm{crit}}(gl_n[t, t^{-1}], c)$. Such elements for $k = 1, \ldots, n$ algebraically generate the center.*

See appendix A.1 for the definitions of $U_{c=\mathrm{crit}}(gl_n[t, t^{-1}], c)$, the Bethe subalgebra and the normal ordering of operators $: \cdots :$.)

5.3. The (quantum) MacMahon–Wronski formula

The following identity for a $s \times s$-matrix M over a commutative ring is called the MacMahon (and sometimes Wronski ([U03, GIOPS98] page 9)) formula:

$$1/\det(1 - M) = \sum_{n=0,\ldots,\infty} \mathrm{Tr}\, S^n M, \tag{56}$$

where $S^n M$ is nth symmetric power of M. It can be easily verified by diagonalizing the matrix M.

Theorem [GLZ03][13] the identity above holds true for Manin matrices (and more generally for their q-analogs) with the following definition of $\mathrm{Tr}\, S^n M$:

$$\mathrm{Tr}\, S^n M = 1/n! \sum_{l_1,\ldots l_n : 1 \leqslant l_i \leqslant n} perm^{\mathrm{row}} \begin{pmatrix} M_{l_1,l_1} & M_{l_1,l_2} & \cdots & M_{l_1,l_n} \\ M_{l_2,l_1} & M_{l_2,l_2} & \cdots & M_{l_2,l_n} \\ \cdots & \cdots & \cdots & \cdots \\ M_{l_n,l_1} & M_{l_n,l_2} & \cdots & M_{l_n,l_n} \end{pmatrix}. \tag{57}$$

We remark that in this formula repeated indices are allowed. The permanent is defined as follows:

$$perm M = perm^{\mathrm{row}} M = \sum_{\sigma \in S_n} \prod_{i=1,\ldots,n} M_{i,\sigma(i)}. \tag{58}$$

This definition of traces of symmetric powers is the same as in the commutative case, with the proviso in mind to use row permanents.

5.3.1. Application. $\mathrm{Tr}\, S^K L(z)$, $ZU_{\mathrm{crit}}(gl_n[t, t^{-1}], c)$, *new Gaudin's integrals.* In this section we apply the quantum MacMahon–Wronski relation to construct further explicit generators in the center of $U_{c=\mathrm{crit}}(gl_n[t, t^{-1}], c)$ and in the commutative Bethe subalgebra in $U(gl_n[t])$), i.e., we give explicit formulae for another set of quantum conservation laws for the quantum Gaudin system. They are quantum analogs of $\mathrm{Tr}\, S^n L^{\mathrm{classical}}(z)$. The elements in the center of $U(gl_n)$ introduced implicitly in [Na91] (the last formula—page 131, see also [U03]) are particular cases of this construction.

Theorem 6. *Let $L(z)$ be a Lax matrix of Gaudin type—see definition 4. Let us define the elements $S_n(z)$ as follows: write $\mathrm{Tr}\, S^n(\partial_z - L(z)) = \sum_{k=0,\ldots,n} c_{k,n}(z)\partial_z^k$ and*

[13] See the papers of Gurevich, Isaev, Ogievetsky, Pyatov and Saponov [GIOPS95, GIOPS05, U03, EP06, HL06, KPak06, FH06] for related results.

set, by definition, $S_n(z) = (-1)^n c_{0,n}(z)$[14]*. These operators commute among themselves and with the coefficients of the quantum characteristic polynomial* $QH_i(z)$*, defined via* $(\det(\partial_z - L(z)) = \sum_i QH_i(z)\partial_z^i)$*. Explicitly*

$$\forall\, k,\, p \text{ and } z, u \in \mathbb{C} : [S_k(z), S_p(u)] = 0, \qquad [S_k(z), QH_p(u)] = 0. \tag{59}$$

Hence they provide new generators among quantum commuting integrals of motion of the Gaudin system.

Corollary 3. *Consider* $L(z) = L_{gl_n[t]}(z)$ *given by formula (lax-glt b). Then* $S_n(z)$ *are generating functions in the variable z for the elements in the commutative Bethe subalgebra in* $U(gl_n[t])$*. Such elements for* $k = 1, \dots, n$ *algebraically generate the Bethe subalgebra.*

Corollary 4. *Consider* $L_{\mathrm{full}}(z)$ *given by formula (A.1); then,* $S_k(z)$ *are generating functions in the variable z for the elements in the center of* $U_{c=\mathrm{crit}}(gl_n[t, t^{-1}], c)$*. Such elements for* $k = 1, \dots, n$ *algebraically generate the center.*

(See appendix A.1 for definitions of $U_{c=\mathrm{crit}}(gl_n[t, t^{-1}], c)$, the Bethe subalgebra and the normal ordering $: \cdots :$.)

Sketch of the proof. The proof follows from the MacMahon–Wronski relations and a theorem of [Ta04] (see formula (2)) on commutativity of coefficients of $\det(\partial_z - L(z))$. Corollary 3 follows immediately, corollary 3 with the help of theorem 7 (see appendix A.1). □

Remark 12. Applying the same construction to the Yangian case, i.e. $\mathrm{Tr}\, S^n\, \mathrm{e}^{-\partial_z} T(z)$ one obtains expressions commuting among themselves and with coefficients of $\det(1 - \mathrm{e}^{-\partial_z} T(z))$. Actually these expressions are quite well known.

6. Concluding remarks and open problems

In this paper, we examined properties of Manin matrices and coherently framed (and generalized) results in the theory of quantum integrable systems within such a perspective.

We think that more work should be done in both of these aspects. We already have some conjectures about new Capelli identities generalizing [Ok96B, MTV06],—in conjunction with the Wick normal ordering of quantum operators, and Baxter-type equations. In this respect (see some preliminary results in appendix A.2) the problem of a good definition and the properties of the quantum Separation of variable scheme seem prominent.

Namely, we deem very important to push further the study of the notion of 'quantum spectral curve' discovered by D Talalaev. In particular, one may hope to describe quantum action-angle variables with some 'quantum Abel–Jacobi transform' related to Talalaev's 'quantum spectral curve'. This might also be important for the general development of noncommutative algebraic geometry. More generally one may hope to develop the methods used in classical integrability: Lax pairs, dressing transformations, tau-functions, explicit soliton and algebro-geometric solutions, etc for quantum systems.

In the influential paper [FFR94] (see also the survey [Fr95]) the relation of quantum Gaudin models with the geometric Langlands correspondence was discovered. We hope that refinements of the techniques discussed herewith be applicable in such a direction, in the so-called local version as well as, possibly, in the global one.

It would also be interesting to define quantum immanants ('fused T-matrices') for $U_{c=\mathrm{crit}}(gl_n[t, t^{-1}], c)$ and to transfer the many remarkable properties established in [OO96]

[14] Namely, the quantities $S_n(z)$ are the 0th order parts of the differential operators $\mathrm{Tr}\, S^n(\partial_z - L(z))$.

for the $U(gl_n)$ case to this more general case. In particular the task is to describe the Harish–Chandra map for $ZU_{c=\mathrm{crit}}(gl_n[t, t^{-1}], c)$. Moving away from the critical level towards (quantum) W-algebras is another task.

We have presented results for gl_n–Gaudin and gl_n–Heisenberg's XXX systems. We plan to extend our study to Q-analogues (i.e. Heisenberg's XXZ system) and super analogue in the future; a paper is in preparation.

The challenge is to consider other systems. For example $\partial_z - L_{so(n)\ or\ sp(n)-\mathrm{Gaudin}}(z)$ is not Manin matrix. However, there are evidences that there should exist an appropriate '$\det(\partial_z - L_{so(n)\ or\ sp(n)-\mathrm{Gaudin}}(z))$', since the existence of such an object is somewhat the core of the Langlands correspondence. The practical problem is, however, to find an explicit and viable formula for the determinant. In this respect we note that, despite $\det^{\mathrm{column}}(\partial_z - L_{so(n)\mathrm{Gaudin}}(z))$ not depending on the order of columns, it does not produce commutative elements already for $so(4)$ (A. Molev, A.C.)). More generally, in view of the existence of a large number of quantum integrable systems, it seems to be natural to address the questions herewith studied to all of them. Namely, we point out the following general question: what are the conditions on a matrix with a noncommutative entry such that there is a natural construction of the determinant and linear algebra works? How efficiently can they be used in the theory of quantum integrable systems? We also plan to come back to these and related questions in the near future.

Acknowledgments

GF acknowledges support from the ESF programme MISGAM, and the Marie Curie RTN ENIGMA (contract number MRTN-CT-2004-5652). The work of AC has been partially supported by the Russian President grant MK-5056.2007.1, the grant of Support for the Scientific Schools 8004.2006.2, by the INTAS grant YSF-04-83-3396, and by the ANR grant GIMP (Geometry and Integrability in Mathematics and Physics). Part of the work was done during the visits of AC to SISSA and the Milano–Bicocca University (under the INTAS project), and to the University of Angers (under ANR grant GIMP). AC is deeply indebted to SISSA and especially to B Dubrovin, as well as to the University of Angers and especially to V Rubtsov, for providing warm hospitality, excellent working conditions and stimulating discussions. The authors are grateful to D Talalaev, M Gekhtman, A Molev, E Sklyanin for multiple stimulating discussions, to P Pyatov for sharing with us his unpublished results, pointing out to the paper [GLZ03] and for multiple stimulating discussions, to S Loktev for his friendly help, to A Smirnov for Maple checks of the conjectures and stimulating discussions, to Yu Burman and N Crampe for their kind remarks.

Appendix

A.1. The center of $U_{c=\mathrm{crit}}(gl_n[t, t^{-1}], c)$, Bethe subalgebras and normal ordering

Let us recall the main result of [CT06] and background (see also [CM]).

$$\text{Consider:}\quad L_{\mathrm{full}}(z) = \sum_{i=-\infty\ldots\infty} \frac{1}{z^{i+1}} \begin{pmatrix} e_{1,1}t^i & \cdots & e_{1,n}t^i \\ \cdots & \cdots & \cdots \\ e_{n,1}t^i & \cdots & e_{n,n}t^i \end{pmatrix}. \tag{A.1}$$

Here, one has to consider $e_{ij}t^k$ as elements of $gl_n[t] \oplus t^{-1}gl_n^{op}[t^{-1}]$,[15] rather than elements of $gl_n[t, t^{-1}]$). In such a case, $L_{\mathrm{full}}(z)$ is of Gaudin type (see definition 4, section 3.1), i.e., it satisfies the relations 9.

[15] By g^{op} is denoted Lie algebra with an opposite commutator: $[g_1, g_2]_{g^{op}} \overset{\mathrm{def}}{=} -[g_1, g_2]_g$.

Via Talalaev's construction one defines the commutative subalgebra in $U(gl_n[t] \oplus t^{-1}gl_n^{op}[t^{-1}])$, i.e. expressions $QH_i(z)$ defined by $\det^{\mathrm{col}}(\partial_z - L_{\mathrm{full}}(z)) = \sum_k \partial_z^k QH_k(z)$ are generating functions in z for some elements in $U\big(gl_n[t] \oplus t^{-1}gl_n^{op}[t^{-1}]\big)$ which commute by Talalaev's theorem and hence generate some commutative subalgebra.

Definition 8. *This commutative subalgebra will be called 'Bethe subalgebra' in* $U\big(gl_n[t] \oplus t^{-1}gl_n^{op}[t^{-1}]\big)$.

Definition 9. *Let us call by 'Bethe subalgebra' in* $U(gl_n[t])$ *the commutative subalgebra defined in a similar way with the help of* $\det^{\mathrm{col}}\big(\partial_z - L_{gl_n[t]}(z)\big)$ *(see (lax-glt b) for* $L_{gl_n[t]}$*).*

This Bethe subalgebra is clearly an image of the Bethe subalgebra in $U(gl_n[t] \oplus t^{-1}gl_n^{op}[t^{-1}])$ under the natural projection $U\big(gl_n[t] \oplus t^{-1}gl_n^{op}[t^{-1}]\big) \to U(gl_n[t])$.

Remark 13. The name 'Bethe subalgebra' was proposed in [NO95] for a related subalgebra in (twisted)-Yangians. (For Yangians (not twisted) the Bethe subalgebra (without a name) was defined in [KS81], formula (5.6), page 114.)

Normal ordering. Let us recall the standard definition of the normal ordering (see, e.g., V Kac [Kac97], formula (2.3.5), page 19, formula (3.1.3), page 37).

Definition 10. *Let* $a(z), b(z)$ *be arbitrary formal power series with values in arbitrary associative algebra (in our case:* $a(z) = (L_{\mathrm{full}}(z))_{ij}, b(z) = (L_{\mathrm{full}}(z))_{kl}$, *for some* i, j, k, l, *(see (A.1) for* $L_{\mathrm{full}}(z)$*). The normally ordered product* $: a(z)b(z) :$ *is defined as follows:*

$$: a(z)b(z) : \stackrel{\mathrm{def}}{=} a(z)_+ b(z) + b(z)a(z)_-, \tag{A.2}$$

where (e.g. V Kac [Kac97], formula (2.3.3), page 19),

$$a(z)_+ \stackrel{\mathrm{def}}{=} \sum_{i\geqslant 0} a_{-i-1}z^i = \sum_{n<0} a_n z^{-n-1}, \qquad a(z)_- \stackrel{\mathrm{def}}{=} \sum_{i<0} a_{-i-1}z^i = \sum_{n\geqslant 0} a_n z^{-n-1}. \tag{A.3}$$

Definition 11. *(e.g. V Kac [Kac97], formula (3.3.1), page 42) The normally ordered product of more than two series* $a^1(z), a^2(z), \ldots, a^n(z)$ *is defined inductively 'from right to left':*

$$: a^1(z)a^2(z)\cdots a^n(z) : \stackrel{\mathrm{def}}{=} : a^1(z) \cdots : a^{n-1}(z)a^n(z) : \cdots : . \tag{A.4}$$

The center of $U_{c=\mathrm{crit}}(gl_n[t, t^{-1}], c)$ *and the 'critical level'*

Let $(gl_n[t, t^{-1}], c)$ be the central extension of the Lie algebra $gl_n[t, t^{-1}]$:

$$[g_1 t^k, g_2 t^l] = [g_1, g_2]t^{k+l} + c(n\,\mathrm{Tr}(g_1 g_2) - \mathrm{Tr}(g_1)\,\mathrm{Tr}(g_2))k\delta_{k,-l}. \tag{A.5}$$

Note that for sl_n the term $\mathrm{Tr}(g_1)\,\mathrm{Tr}(g_2)$ disappears and this is the standard central extension up to normalization.

Fact [Ha88, GW89, FF92] The center of $U_{c=\kappa}(gl_n[t, t^{-1}], c)$[16] is trivial: $\mathbb{C} \times 1$, unless $\kappa \neq -1$. For $\kappa = -1$ there exists a large center. $c = -1$ is called 'critical level'.

Theorem 7.

$$: \det^{\mathrm{col}}(L_{\mathrm{full}}(z) - \partial_z) : \text{ generates the center on the critical level} \tag{A.6}$$

i.e. define the elements H_{ij} *by* $: \det^{\mathrm{col}}(L_{\mathrm{full}}(z) - \partial_z) := \mathrm{sum}_{i=-\infty\cdots\infty;\, j=0\ldots n} H_{ij} z^i \partial_z^j$, *then* H_{ij} *freely generates the center of* $U_{c=-1}(gl_n[t, t^{-1}], c)$. *Here we consider* $L_{\mathrm{full}}(z)$ *(see (A.1)) as a matrix-valued generating function for generators of* $U_{c=-1}(gl_n[t, t^{-1}], c)$, *i.e.* $e_{mn}t^p \in U_{c=-1}(gl_n[t, t^{-1}], c)$.

[16] More precisely one needs 'local completion' of $U_{c=\kappa}(gl_n[t, t^{-1}], c)$, see [FFR94].

The theorem was first proved in [CT06], but the definition of the normal ordering given there is different. The coincidence with the standard normal ordering is due to [CM], where another direct proof of the theorem will be given. The proof in [CT06] is quite short, but requires various results: Talalaev's theorem, existence of the center, AKS-arguments, remarkable ideas from [Ry1, Ry3]. In [CM] a direct proof will be given, as a corollary new proof of Talalaev's theorem will be obtained.

A.2. Quantum separation of variables

Let us briefly discuss some results and conjectures about the problem of separation of variables. E Sklyanin (see the surveys [Sk92, Sk95]) proposed an approach which potentially should give the most powerful way to solve integrable systems. This program is far from being completed. The ultimate goal in this framework is to construct coordinates α_i, β_i such that a joint eigenfunction of all Hamiltonians will be presented as product of functions of one variable:

$$\Psi(\beta_1, \beta_2, \ldots) = \prod_i \Psi^{1-\text{particle}}(\beta_i).$$

We consider this construction at the quantum level, trying to extend in this realm some ideas of Sklyanin and [AHH93, DD94, Ge95] and others.

We have checked the validity of the conjectures below for 2×2 and 3×3 cases (in particular comparing with the results of [Sk92b]). Let us recall that, in the classical case, the construction of separated variables for the systems we are considering goes, somewhat algorithmically[17], as follows:

Step 1. One considers, for a gl_n model, along the Lax matrix $L(z)$, the matrix $M = \lambda - L(z)$ and its classical adjoint $M^\vee$.

Step 2. One takes a vector ψ by means of suitable linear combination of columns (or rows) of $M^\vee$; in the simplest case, one can take ψ to be one of the columns, say the last of $M^\vee$. One seeks for pairs (λ_i, z_i) that solve

$$\psi_i = 0, \quad i = 1, \ldots, n.$$

Step 3. To actually solve this problem, one proceeds as follows. As each component ψ_i of ψ is a polynomial of degree at most $n-1$, one can form, out of ψ the matrix M_ψ, collecting the coefficients of the expansion of ψ_i in powers of λ, i.e.:

$$[M_\psi]_{j,i} = \mathrm{res}_{\lambda=0}\psi_i\lambda^{n-j-1}, \qquad i = 1, \ldots, n, \quad j = 1, \ldots, n.$$

Step 4. The separation coordinates are given by pairs (λ_i, z_i) where z_i's are roots of $\mathrm{Det}(M_\psi)$, and λ_i are the corresponding values of λ_i[18], that can be obtained, e.g., via the Cramer's rule. By construction , the Jacobi separation relations are the equation(s) of the spectral curve, $\mathrm{Det}(\lambda - L(z) = 0$.

The Yangian case, $n = 2, 3$. Let $T(z)$ be a Lax matrix of the Yangian type (see section 3.2), so $(1 - \mathrm{e}^{-\partial_z}T(z))$ is a Manin matrix and an adjoint matrix can be calculated by standard formulae (see section 4.1). Consider:

$$(1 - \mathrm{e}^{-\partial_z}T(z))^{\text{adjoint}} = \begin{pmatrix} (1 - \mathrm{e}^{-\partial_z}T(z))^{\text{adjoint}}_{1,1} & \cdots & (1 - \mathrm{e}^{-\partial_z}T(z))^{\text{adjoint}}_{1,n} \\ \cdots & \cdots & \cdots \\ (1 - \mathrm{e}^{-\partial_z}T(z))^{\text{adjoint}}_{n,1} & \cdots & (1 - \mathrm{e}^{-\partial_z}T(z))^{\text{adjoint}}_{n,n} \end{pmatrix}. \tag{A.7}$$

[17] We are herewith sweeping under the rug the problem known as 'normalization of the Baker Akhiezer function'.

[18] In the quadratic R-matrix case, actually one has to take as canonical momenta, the logarithms of these λ_i.

Let us take an arbitrary column of this matrix, say the last column, and denote by $M_{i,j}(z)$ the following:

$$\begin{pmatrix} (1-e^{-\partial_z}T(z))^{\text{adjoint}}_{1,n} \\ \cdots \\ (1-e^{-\partial_z}T(z))^{\text{adjoint}}_{n,n} \end{pmatrix} = \begin{pmatrix} \sum_{k=0\ldots n-1} e^{-k\partial_z} M_{1,k}(z) \\ \cdots \\ \sum_{k=0\ldots n-1} e^{-k\partial_z} M_{n,k}(z) \end{pmatrix}. \tag{A.8}$$

In other words, M_{ij} is the matrix of the coefficients of expansion in left powers of $e^{-\partial_z}$ of the elements of the last column of $(1-e^{-\partial_z}T(z))^{\text{adjoint}}$.

Let us call 'M-matrix' the matrix of these coefficients, that is:

$$M_{n\times n}(z) \stackrel{\text{def}}{=} \begin{pmatrix} M_{1,0}(z) & \cdots & M_{n-1,0}(z) \\ \cdots & \cdots & \cdots \\ M_{n,0}(z) & \cdots & M_{n-1,n}(z) \end{pmatrix}. \tag{A.9}$$

In the cases of low matrix rank (i.e., $n = 2, 3$), computations can be done explicitly. The following arguments hold.

(1) Define $B(z) = Det^{\text{column}}(M(z))$; then

$$[B(z), B(u)] = 0. \tag{A.10}$$

(2) Consider any root β of the equation $B(u) = 0$, (it belongs to an appropriate algebraic extension of the original noncommutative algebra R). Then the overdetermined system of, respectively, equations (2) and (3) for the single variable α has a unique solution:

$$\begin{pmatrix} M_{1,0}(z)|_{\text{Substitute left } z\to\beta} & \cdots & M_{1,n-1}(z)|_{\text{Substitute left } z\to\beta} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ M_{n,0}(z)|_{\text{Substitute left } z\to\beta} & \cdots & M_{n,n-1}(z)|_{\text{Substitute left } z\to\beta} \end{pmatrix} \begin{pmatrix} \alpha^{n-1} \\ \alpha^{n-2} \\ \cdots \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ .. \\ 0 \end{pmatrix}. \tag{A.11}$$

(3) Consider all the roots β_i of the equations $B(u) = 0$, and the corresponding variables α_i. Then:

- the variables α_i, β_i satisfy the following commutation relations:

$$[\alpha_i, \beta_j] = -\alpha_i\delta_{i,j}, \tag{A.12}$$

$$[\alpha_i, \alpha_j] = 0, \qquad [\beta_i, \beta_j] = 0, \tag{A.13}$$

- α_i, β_i satisfy the 'quantum characteristic equation':

$$\forall\, i: \det(1-e^{-\partial_z}T(z))|_{\text{Substitute left } z\to\beta_i;e^{-\partial_z}\to\alpha_i} = 0 \tag{A.14}$$

- if $T(z)$ is generic, then variables α_i, β_i are 'quantum coordinates' i.e. all elements of the algebra R can be expressed via α_i, β_i and the centre (Casimirs) of the algebra R.

The proof of these statements can be done by direct calculations; we remark that our formulae reproduce those of the paper [Sk92b].

Conjecture. *It is natural to conjecture that the same hold for higher values of n, that is, that the M matrix M, and it provides a quantum separated variable for the $gl(n)$ Yangian case.*

Remark 14. In the classical case (and also for the Gaudin model) this solution of the separation of variables problem can be explicitly found in [AHH93]. The Poisson version of the conjecture above about Manin properties of Yangian systems was not yet, to the best of our knowledge, considered in the literature; however, it can possibly be deduced from the results of [DD94, Ge95].

Remark 15. An open problem is a corresponding conjecture for the Gaudin case. The conjecture holds true for $n = 2$, but seems not directly extendable to the case $n > 2$.

References

[AHH93] Adams M R, Harnad J and Hurtubise J 1993 Darboux coordinates and Liouville–Arnold integration in Loop algebrs *Commun. Math. Phys.* **155** 385–413 http://projecteuclid.org/

[BBT03] Babelon O, Bernard D and Talon M 2003 *Introduction to Classical Integrable Systems* (Cambridge: Cambridge University Press)

[BT07] Babelon O and Talalaev D 2007 On the Bethe Ansatz for the Jaynes–Cummings–Gaudin model *Preprint* hep-th/0703124

[BT02] Babelon O and Talon M 2002 Riemann surfaces, separation of variables and classical and quantum integrability *Preprint* hep-th/0209071

[CFRy07] Chervov A, Falqui G and Rybnikov L 2007 Limits of Gaudin algebras, quantization of bending flows, Jucys–Murphy elements and Gelfand–Tsetlin bases (*Preprint* 0710.4971)

[CF69] Cartier P and Foata D 1969 *Problemes Combinatoires de Commutation et Rearrangements (Lecture Notes in Mathematics* vol 85*)* (Berlin: Springer)

[CFRu] Chervov A, Falqui G and Rubtsov V in preparation

[CM] Chervov A and Molev A in preparation

[CRyT04] Chervov A, Rybnikov L and Talalaev D 2004 Rational Lax operators and their quantization *Preprint* hep-th/0404106

[CT04] Chervov A and Talalaev D 2004 Universal G-oper and Gaudin eigenproblem *Preprint* hep-th/0409007

[CT06] Chervov A and Talalaev D 2006 Quantum spectral curves, quantum integrable systems and the geometric Langlands correspondence *Preprint* hep-th/0604128

[CT06b] Chervov A and Talalaev D 2006 KZ equation, G-opers, quantum Drinfeld–Sokolov reduction and Cayley–Hamilton identity *Preprint* hep-th/0607250

[Di03] Dickey L A 2003 *Soliton Equations and Hamiltonian Systems (Advanced Series in Mathematical Physics* vol 26*)* 2nd edn (River Edge, NJ: World Scientific) xii+408 pp, ISBN: 981-238-173-2

[Dr85] Drinfeld V 1985 Hopf Algebras and the quantum Yang–Baxter equation *Sov. Math. Dokl.* **32** 254–8

[DD94] Dubrovin B and Diener P 1994 Algebro-geometrical Darboux coordinates in R-matrix formalism, SISSA-88/94FM, CERN Libraries SCAN-9409240

[ER01] Enriquez B and Rubtsov V 2001 Commuting families in skew fields and quantization of Beauville's fibration *Preprint* math.AG/0112276

[EP06] Pavel E and Igor P 2006 An algebraic extension of the MacMahon master theorem *Preprint* math/0608005

[FT79] Faddeev L D, Takhdadjan L A and Sklyanin E K 1979 Quantum inverse problem method: I *Theor. Math. Phys.* **40** N2 http://www.springerlink.com/

[FF92] Feigin B and Frenkel E 1992 *Int. J. Mod. Phys.* A **7** 197–215

[FFR94] Feigin B, Frenkel E and Reshetikhin N 1994 *Commun. Math. Phys.* **166** 27–62 (*Preprint* hep-th/9402022)

[Fo79] Foata D 1979 A noncommutative version of the matrix inversion formula *Adv. Math.* **31** 330–49

[FH06] Foata D and Han G N 2006 A basis for the right quantum algebra and the '$1 = q$' principle *Preprint* math/0603463

[Fr95] Frenkel E 1995 Affine algebras, Langlands duality and Bethe Ansatz *Preprint* q-alg/9506003

[GIOPS05] Gurevich D, Pyatov P and Saponov P 2005 The GL(m–n) type quantum matrix algebras II: the structure of the characteristic subalgebra and its spectral parameterization *Preprint* math.QA/0508506

[GIOPS95] Pyatov P and Saponov P 1995 Characteristic relations for quantum matrices (*Preprint* q-alg/9502012)

[GIOPS98] Isaev A, Ogievetsky O and Pyatov P 1998 On quantum matrix algebras satisfying the Cayley–Hamilton–Newton identities (*Preprint* math/9809170)

[GLZ03] Garoufalidis S, Le T T and Zeilberger D 2003 The quantum MacMahon Master Theorem (*Preprint* math.QA/0303319)

[Ga76] Gaudin M 1976 Diagonalisation d' une classe d' hamiltoniens de spin *J. Phys.* **37** 1087–98

[Ga83] Gaudin M 1983 *La Fonction d' Onde de Bethe* (Paris: Masson)

[Ge95] Gekhtman M I 1995 Separation of variables in the classical SL(N) magnetic chain *Commun. Math. Phys.* **167** 593–605 http://projecteuclid.org/

[GR97] Gelfand I and Retakh V 1997 Quasideterminants: I (*Preprint* q-alg/9705026)

[GGRW02] Gelfand I, Gelfand S, Retakh V and Wilson R 2002 Quasideterminants (*Preprint* math.QA/0208146)

[GW89] Goodman R and Wallach N 1989 Higher-order Sugawara operators for affine Lie algebras *Trans. Am. Math. Soc.* **315** 1–55 http://links.jstor.org/sici?sici=0002-9947(198909)315%3A1%3C1%3AHSOFAL%3E2.0.CO%3B2-S

[HL06] Ho H P and Martin L 2006 Koszul algebras and the quantum MacMahon master theorem (*Preprint* math/0603169)

[Ha88] Hayashi T 1988 Sugawara operators and Kac–Kazhdan conjecture *Invent. Math.* **94** 13–52 http://www.springerlink.com/content/wp5nwm73g35147u5/

[Kac97] Victor K 1997 *Vertex Algebras for Beginners (University Lecture Series* vol 10*)* (Providence, RI: American Mathematical Society) viii+141 pp

[Kir00] Kirillov A A 2001 Introduction to family algebras *Mosc. Math. J.* **1** 49–63 http://www.ams.org/distribution/mmj/vol1-1-2001/kirillov.pdf

[KZ84] Knizhnik V G and Zamolodchikov A B 1984 Current algebra and Wess–Zumino model in two-dimensions *Nucl. Phys.* B **247** 83–103 http://dx.doi.org/10.1016/0550-3213(84)90374-2

[K97] Kontsevich M 1997 Deformation quantization of Poisson manifolds, I (*Preprint* q-alg/9709040)

[Ko07A] Konvalinka M 2007 A generalization of Foata's fundamental transformation and its applications to the right-quantum algebra (*Preprint* math.CO/0703203)

[Ko07B] Konvalinka M 2007 Non-commutative Sylvester's determinantal identity (*Preprint* math.CO/0703213)

[KPak06] Matjaz K and Igor P 2007 Non-commutative extensions of the MacMahon master theorem (*Preprint* math/0607737)

[KL94] Krob D and Leclerc B 1995 Minor identities for quasi-determinants and quantum determinants *Commun. Math. Phys.* **169** 1–23 (*Preprint* q-alg/9411194)

[KS81] Kulish P P and Sklyanin E K 1981 Quantum spectral transform method recent developments *Integrable Quantum Field Theories: Proc. Symp. (Tvarminne, Finland, 23–27 March) (Lecture Notes in Physics* vol 151*)* ed J Hietarinta and C Montonen pp 61–119 http://www.springerlink.com/content/750u2k2xg12w8886/

[Man87] Manin Y E 1987 Some remarks on Koszul algebras and quantum groups *Ann. l'institut Fourier* **37** 191–205 http://www.numdam.org/

[Man91] Manin Y 1991 *Topics in Non Commutative Geometry* (Princeton, NJ: Princeton University Press)

[Man88] Manin Y 1988 *Quantum Groups and Non Commutative Geometry* (Centre De Recherches Mathematiques)

[Mol02] Molev A I 2002 Yangians and their applications (*Preprint* math.QA/0211288)

[MNO94] Molev A, Nazarov M and Olshanski G 2002 Yangians and classical Lie algebras (*Preprint* math.QA/0211288)

[MTV05] Mukhin E, Tarasov V and Varchenko A 2005 The B. and M. Shapiro conjecture in real algebraic geometry and the Bethe ansatz (*Preprint* math.AG/0512299)

[MTV06] Mukhin E, Tarasov V and Varchenko A 2006 A generalization of the Capelli identity (*Preprint* math.QA/0610799)

[Na91] Nazarov M 1991 Quantum Berezinian and the classical capelli identity *Lett. Math. Phys.* **21** http://www.springerlink.com/content/rt5r3313732p48j7

[NO95] Nazarov M and Olshanski G 1995 Bethe subalgebras in twisted Yangians (*Preprint* q-alg/9507003)

[Na96] Nazarov M L 1998 Yangians and Capelli identities *Kirillov's Seminar on Representation Theory Am. Math. Soc. Transl., Ser. 2* vol 181*)* ed G I Olshanski (Providence, RI: American Mathematical Society) pp 139–63 (*Preprint* q-alg/9601027)

[Ok96B] Okounkov A 1996 Young basis, Wick formula and higher Capelli identities (*Preprint* q-alg/9602027)

[OO96] Okounkov A and Olshansky G 1997 *Algebra Anal.* **9** 73–146 (Russian) (*Preprint* q-alg/9605042)
Okounkov A and Olshansky G 1998 Shifted Schur Functions *St. Petersburg Math. J.* **9** (Eng. transl.)

[Re92] Reshetikhin N 1992 The Knizhnik-Zamolodchikov system as a deformation of the isomonodromy problem *Lett. Math. Phys.* **26** 167–77 http://www.springerlink.com/content/t718750378552740

[FRT90] Reshetikhin N Yu, Takhtadzhyan L A and Faddeev L D 1989 *Algebra Anal.* **1** 178–206
Reshetikhin N Yu, Takhtadzhyan L A and Faddeev L D 1990 Quantization of Lie groups and Lie algebras (English. Russian original) *Leningr. Math. J.* **1** 193–225 (Translation)

[Ry1] Rybnikov L G 2006 *Funktsional'nyi Analiz i Ego Prilozheniya* **40** 30–43 (*Preprint* math.RT/0606380)

[Ry3] Rybnikov L G 2006 Argument shift method and Gaudin model *Func. Anal. Appl.* **40** (Eng. Transl.)
Rybnikov L G 2006 Uniqueness of higher Gaudin hamiltonians (*Preprint* math.QA/0608588)

[Sa06] Sakhnovich L 2006 Rational solutions of Knizhnik–Zamolodchikov system(*Preprint* math-ph/0609067)

[Sk92] Sklyanin E K 1992 Quantum inverse scattering method. Selected topics (*Preprint* hep-th/9211111)

[Sk92b] Sklyanin E K 1992 Separation of variables in the quantum integrable models related to the Yangian Y[sl(3)] (*Preprint* hep-th/9212076)

[Sk95] Sklyanin E K 1995 Separations of variables: new trends *Progr. Theor. Phys. Suppl.* **118** 35–60 (*Preprint* solv-int/9504001)

[Ta04] Talalaev D 2006 Quantization of the Gaudin system *Funct. Anal. Its Appl.* **40** 86–91 (*Preprint* hep-th/0404153)

[U03] Umeda T 2003 Application of Koszul complex to Wronski relations for $U(\mathfrak{gl}_n)$ *Commentarii Math. Helvetici* **78** 663–80 http://www.springerlink.com/content/2l45hauppwnjc6be/